

# Monotonicity formula and Liouville-type theorems of stable solution for the weighted elliptic system\*

Liang-Gen Hu<sup>†</sup>

Department of Mathematics, Ningbo University, 315211, P.R. China

**Abstract:** In this paper, we are concerned with the weighted elliptic system

$$\begin{cases} -\Delta u = |x|^\beta v^\vartheta, \\ -\Delta v = |x|^\alpha |u|^{p-1} u, \end{cases} \quad \text{in } \Omega,$$

where  $\Omega$  is a subset of  $\mathbb{R}^N$ ,  $N \geq 5$ ,  $\alpha > -4$ ,  $0 \leq \beta \leq \frac{N-4}{2}$ ,  $p > 1$  and  $\vartheta = 1$ .

We first apply Pohozaev identity to construct a monotonicity formula and reveal their certain equivalence relation. By the use of *Pohozaev identity*, *monotonicity formula* of solutions together with a *blowing down* sequence, we prove Liouville-type theorems of stable solutions (whether positive or sign-changing) for the weighted elliptic system in the higher dimension.

**Keywords:** Liouville-type theorem; stable solutions; Pohozaev identity; monotonicity formula; blowing down sequence

## 1 Introduction

In this article, we examine the nonexistence of classical stable solutions of the weighted elliptic system given by

$$\begin{cases} -\Delta u = |x|^\beta v^\vartheta, \\ -\Delta v = |x|^\alpha |u|^{p-1} u, \end{cases} \quad \text{in } \Omega, \quad (1.1)$$

where  $\Omega$  is a subset of  $\mathbb{R}^N$ ,  $N \geq 5$ ,  $\alpha > -4$ ,  $0 \leq \beta \leq \frac{N-4}{2}$  and  $p\theta > 1$ .

---

\*The work was partially supported by NSFC of China (No. 11201248), K.C. Wong Fund of Ningbo University and Ningbo Natural Science Foundation (No. 2014A610027).

<sup>†</sup>email address: hulianggen@tom.com

The idea of using the Morse index of a solution for a semilinear elliptic equation was first explored by Bahri and Lions [1] to get further qualitative properties of the solution. Recently, along this line of research, Dancer [4–6] introduced the finite Morse index solution and made the significant progress in the elliptic equations. Let us note that the solution  $u$  is stable if and only if its Morse index is equal to zero. In 2007, Farina considered the Lane-Emden equation

$$-\Delta u = |u|^{p-1}u, \quad (1.2)$$

on bounded and unbounded domains of  $\Omega \subset \mathbb{R}^N$ , with  $N \geq 2$  and  $p > 1$ . Based on a delicate application of the classical Moser's iteration, he gave the complete classification of finite Morse index solutions (positive or sign-changing) in his seminal paper [14]. Hereafter, many experts utilized the Moser's iterative method to discuss the stable and finite Morse index solutions of the harmonic and fourth-order elliptic equation and obtained many excellent results. We refer to [7, 29, 31, 32] and the reference therein.

However, the classical Moser's iterative technique does not completely classify finite Morse index solutions of the biharmonic equation

$$\Delta^2 u = |u|^{p-1}u, \quad \text{in } \Omega \subset \mathbb{R}^N.$$

To solve the problem, Dávila et al. [9] have recently derived a monotonicity formula of solutions and given the complete classification of stable and finite Morse index solutions for the biharmonic equation by the application of Pohozaev identity and the monotonicity formula. We note that many outstanding papers [8, 9, 20, 21, 30] utilize a monotonicity formula to study the partial regularity of stationary weak solution, stable and finite Morse index solutions for the harmonic and fourth-order equation.

On the other hand, some experts were interesting in the Lane-Emden system and obtained some excellent results [3, 11–13]. In 2013, applying a iterative method and the pointwise estimate in [28], Cowan proved the following result.

**Theorem A.** ([3, Theorem 2]) *Suppose that  $p > \theta = 1$ ,  $\alpha = \beta = 0$  and*

$$N < 2 + \frac{4(p+1)}{p-1} \left( \sqrt{\frac{2p}{p+1}} + \sqrt{\frac{2p}{p+1}} - \sqrt{\frac{2p}{p+1}} \right).$$

*Then there is no positive stable solution of (1.1).*

Adopting the same method as Cowan [3], Fazly obtained the following result.

**Theorem B.** ([12, Theorem 2.4]) *Suppose that  $(u, v)$  is  $C^2(\mathbb{R}^N)$  nonnegative entire semi-stable solution of*

$$\begin{cases} -\Delta u = \rho(1 + |x|^2)^{\frac{\alpha}{2}} v, \\ -\Delta v = \varrho(1 + |x|^2)^{\frac{\alpha}{2}} u^p, \end{cases}$$

*with  $\rho, \varrho > 0$  in the dimension*

$$N < 8 + 3\alpha + \frac{8 + 4\alpha}{p - 1}.$$

*Then,  $(u, v)$  is the trivial solution.*

We observe that the dimension  $N < 8 + 3\alpha + \frac{8 + 4\alpha}{p - 1}$  in [12, Theorem 2.4] is already larger than the *critical hyperbola*, i.e.,  $N = 4 + \alpha + \frac{8 + 4\alpha}{p - 1}$ . Recently, Fazly and Ghoussoub [13, Theorem 4] have considered the nonexistence of positive stable solutions for the weighted elliptic system (1.1), which the dimension satisfies

$$N < 2 + 2 \left( \frac{p(\beta + 2) + \alpha + 2}{p\theta - 1} \right) \left( \sqrt{\frac{p\theta(\theta + 1)}{p + 1}} + \sqrt{\frac{p\theta(\theta + 1)}{p + 1} - \sqrt{\frac{p\theta(\theta + 1)}{p + 1}}} \right).$$

Clearly, if  $\theta = 1$  and  $\alpha = \beta = 0$  in (1.1), then their result is the same as Theorem A.

Let us briefly recall the fact that Liouville-type theorem of solutions for various Lane-Emden equations and systems is interesting and challenging for decades.

First, Pohozaev identity shows that the Lane-Emden equation with the Dirichlet boundary condition has no positive solution on a bounded star-shaped domain  $\Omega \subset \mathbb{R}^N$ , whenever  $p \geq \frac{N + 2}{N - 2}$ . On the other hand, Gidas and Spruck obtained the optimal Liouville-type theorems in the celebrated paper [16], that is, the Lane-Emden equation (1.2) has no positive solution if and only if  $1 < p < \frac{N + 2}{N - 2}$  ( $= +\infty$ , if  $N \leq 2$ ). In 1991, Bidaut-Véron and Véron [2] obtained the asymptotic behavior of positive solution by utilizing the Bochner-Lichnerowicz-Weitzenböck formula in  $\mathbb{R}^N$ .

In the case of the Lane-Emden systems (1.1) with  $\alpha = \beta = 0$ , Pucci and Serrin [25] proved that if  $\frac{N}{p + 1} + \frac{N}{\theta + 1} \leq N - 2$  and  $\Omega$  is a bounded star-shaped domain of  $\mathbb{R}^N$ , then there is no positive solution of (1.1) with the Dirichlet boundary conditions. Noting that the curve  $\frac{N}{p + 1} + \frac{N}{\theta + 1} = N - 2$  is the *critical Sobolev hyperbola*. Similar to the Lane-Emden equation, the following conjecture is interesting and challenging.

**Conjecture (Lane-Emden Conjecture)** *Suppose  $(p, \theta)$  is under the critical Sobolev hy-*

perbola, i.e.,

$$\frac{N}{p+1} + \frac{N}{\theta+1} > N - 2.$$

Then there is no positive solution for the elliptic system (1.1) with  $\alpha = \beta = 0$ .

The case of radial solutions was solved by Mitidieri [18] in any dimension, and the positive radial solutions on and above the critical Sobolev hyperbola was constructed by [18, 27], which is the optimal Liouville-type theorem for radial solutions. The *conjecture* (for non-radial solutions) seems difficult. In the dimension  $N = 3$ , Serrin and Zou [26] proved the *conjecture* for the polynomially bounded solutions, which the boundedness was removed in [24]. In 2009, Souplet [28] solved the *conjecture* in  $N = 4$  or a new region for  $N \geq 5$ . However, the weighted Lane-Emden system (1.1) is even less understood. For example, the paper [23] proved the *conjecture* for the equation  $-\Delta u = |x|^\alpha u^p$  in  $N = 3$ ; In 2012, Phan [22] solved the *conjecture* for the system (1.1) in two cases: *case 1.*  $N = 3$  and bounded solutions; *case 2.*  $N = 3$  or 4 and  $\alpha, \beta \leq 0$ .

Here and in the following, we always assume that  $N \geq 5$ ,  $\alpha > -4$ ,  $0 \leq \beta \leq \frac{N-4}{2}$ ,  $p > 1$  and  $\theta = 1$ . Motivated by the ideas in [9, 10, 17], we will construct a monotonicity formula of solutions in the dimension  $4 + \beta + \frac{8+2\alpha+2\beta}{p-1} < N < N_{\alpha,\beta}(p)$  ( $N_{\alpha,\beta}(p)$  see below (3.1)) and get various integral estimates, and then use these results to study Liouville-type theorems of stable solution for the weighted elliptic system (1.1).

**Theorem 1.1.** *For any  $4 + \beta + \frac{8+2\alpha+2\beta}{p-1} < N < N_{\alpha,\beta}(p)$ , assume that  $u \in W_{loc}^{2,2}(\mathbb{R}^N \setminus \{0\})$  is a homogeneous, stable solution of (1.1),  $|x|^\alpha |u|^{p+1} \in L_{loc}^1(\mathbb{R}^N \setminus \{0\})$  and  $|x|^{-\beta} |\Delta u|^2 \in L_{loc}^1(\mathbb{R}^N \setminus \{0\})$ . Then  $u \equiv 0$ .*

Applying Theorem 1.1 and the properties of monotonicity formula (2.14), we get

**Theorem 1.2.** *If  $u \in C^4(\mathbb{R}^N)$  is a stable solution of (1.1) in  $\mathbb{R}^N$  and  $5 \leq N \leq N_{\alpha,\beta}(p)$ , then  $u \equiv 0$ .*

**Remark 1.1.** (1) *We apply Pohozaev identity to construct a monotonicity formula.*

*From the process of the proof in Theorem 2.1, we can observe that Pohozaev identity is equivalence to the certain derivative-type of the monotonicity formula.*

(2) *Let us note that for the dimensions  $4 + \beta + \frac{8+2\alpha+2\beta}{p-1} < N < N_{\alpha,\beta}(p)$ , we adopt a new method of monotonicity formula together with blowing down sequence to investigate Liouville-type theorem. In addition, a difficulty stems from the fact*

that the terms  $|x|^\alpha$  and  $|x|^\beta$  in (1.1) leads to the singularity. For this reason, we use a more delicate approach to derive improved integral estimates.

(3) From the computation of  $N_{\alpha,\beta}(p)$  (in Section 3), we find the following relation:

$$\begin{cases} N_{0,0}(p) > 2 + \frac{4(p+1)}{p-1} \left( \sqrt{\frac{2p}{p+1}} + \sqrt{\frac{2p}{p+1}} - \sqrt{\frac{2p}{p+1}} \right), & \text{if } \alpha = \beta = 0, \\ N_{\alpha,\alpha}(p) > 8 + 3\alpha + \frac{8+4\alpha}{p-1}, & \text{if } \alpha = \beta. \end{cases}$$

Therefore, in contrast with **Theorem A** and **Theorem B**, we obtain Liouville-type theorem in the higher dimension.

Next, we list some definitions and notations. Let  $\Omega$  be a subset of  $\mathbb{R}^N$  and  $f, g \in C^1(\mathbb{R}^{N+2}, \Omega)$ . Following Montenegro [19], we consider the general elliptic system

$$(S_{f,g}) \begin{cases} -\Delta u = f(u, v, x), \\ -\Delta v = g(u, v, x), \end{cases} \quad x \in \Omega.$$

A solution  $(u, v) \in C^2(\Omega) \times C^2(\Omega)$  of  $(S_{f,g})$  is called *stable*, if the eigenvalue problem

$$(E_{f,g}) \begin{cases} -\Delta \phi = f_u(u, v, x)\phi + f_v(u, v, x)\psi + \eta\phi, \\ -\Delta \psi = g_u(u, v, x)\phi + g_v(u, v, x)\psi + \eta\psi, \end{cases}$$

has a first positive eigenvalue  $\eta > 0$ , with corresponding positive smooth eigenvalue pair  $(\phi, \psi)$ . A solution  $(u, v)$  is said to be *semi-stable*, if the first eigenvalue  $\eta$  is nonnegative.

Inspired by the above definition, we give the integration-type definition of stability.

**Definition 1.1.** We recall that a critical point  $u \in C^4(\Omega)$  of the energy function

$$\mathcal{E}(u) = \int_{\Omega} \left[ \frac{1}{2} \frac{|\Delta u|^2}{|x|^\beta} - \frac{1}{p+1} |x|^\alpha |u|^{p+1} \right] dx$$

is said to be a *stable solution* of (1.1), if, for any  $\zeta \in C_0^2(\Omega)$ , we have

$$p \int_{\Omega} |x|^\alpha |u|^{p-1} \zeta^2 dx \leq \int_{\Omega} \frac{|\Delta \zeta|^2}{|x|^\beta} dx.$$

The definition is interesting and well-defined. In deed, if  $(u, v)$  is a semi-stable solution, then there exist  $\eta \geq 0$  and a positive smooth eigenvalue pair  $(\phi, \psi)$  such that

$$\begin{cases} -\Delta \phi = |x|^\beta \psi + \eta\phi, \\ -\Delta \psi = p|x|^\alpha |u|^{p-1} \phi + \eta\psi. \end{cases}$$

Multiply the second equation by  $\frac{\zeta^2}{\phi}$  with  $\zeta \in C_0^2(\Omega)$  to get

$$\begin{aligned} p \int_{\Omega} |x|^{\alpha} |u|^{p-1} \zeta^2 dx &\leq \int_{\Omega} -\Delta \psi \frac{\zeta^2}{\phi} dx = \int_{\Omega} -\psi \Delta \left( \frac{\zeta^2}{\phi} \right) dx \\ &= \int_{\Omega} \frac{1}{|x|^{\beta}} \left[ 1 - \frac{\eta \phi}{|x|^{\beta} \psi + \eta \phi} \right] \Delta \phi \Delta \left( \frac{\zeta^2}{\phi} \right) dx. \end{aligned} \quad (1.3)$$

A simple calculation leads to

$$\Delta \left( \frac{\zeta^2}{\phi} \right) = 2\phi^{-1} |\nabla \zeta|^2 + 2\zeta \phi^{-1} \Delta \zeta - 4\zeta \phi^{-2} \nabla \zeta \cdot \nabla \phi + 2\zeta^2 \phi^{-3} |\nabla \phi|^2 - \zeta^2 \phi^{-2} \Delta \phi.$$

Then we find

$$\begin{aligned} \Delta \phi \Delta \left( \frac{\zeta^2}{\phi} \right) - |\Delta \zeta|^2 &= 2\zeta \phi^{-1} \Delta \zeta \Delta \phi - \zeta^2 \phi^{-2} |\Delta \phi|^2 - |\Delta \zeta|^2 \\ &\quad + 2\phi^{-1} \Delta \phi [|\nabla \zeta|^2 - 2\zeta \phi^{-1} \nabla \zeta \cdot \nabla \phi + \zeta^2 \phi^{-2} |\nabla \phi|^2] \\ &= - \left[ (\zeta \phi^{-1} \Delta \phi - \Delta \zeta)^2 + 2(\phi^{-1} |x|^{\beta} \psi + \eta) (\nabla \zeta - \zeta \phi^{-1} \nabla \phi)^2 \right] \\ &\leq 0, \end{aligned}$$

implies

$$\int_{\Omega} \frac{\Delta \phi}{|x|^{\beta}} \Delta \left( \frac{\zeta^2}{\phi} \right) dx \leq \int_{\Omega} \frac{|\Delta \zeta|^2}{|x|^{\beta}} dx.$$

Therefore, combining the above inequality with (1.3), we obtain

$$p \int_{\Omega} |x|^{\alpha} |u|^{p-1} \zeta^2 dx \leq \int_{\Omega} \frac{|\Delta \zeta|^2}{|x|^{\beta}} dx.$$

**Remark 1.2.** Since  $\phi$  is a smooth function,  $\zeta \in C_0^2(\Omega)$  and  $\beta \leq \frac{N-4}{2}$ , then the integration  $\int_{\Omega} \frac{1}{|x|^{\beta}} dx$  is well defined.

**Notations.** Throughout this paper,  $B_r(x)$  denotes the open ball of radius  $r$  centered at  $x$ . If  $x = 0$ , we simply denote  $B_r(0)$  by  $B_r$ .  $C$  denotes various irrelevant positive constants.

The rest of the paper is organized as follows. In Section 2, we derive various integral estimates and construct a monotonicity formula. In Section 3, we prove Liouville-type theorem of homogeneous, stable solutions in the dimensions  $4 + \beta + \frac{8 + 2\alpha + 2\beta}{p-1} < N < N_{\alpha, \beta}(p)$ . Finally, we study the qualitative properties of the monotonicity function  $\mathcal{M}$ , and prove Theorem 1.2 which is based on *Pohozaev-type identity*, *monotonicity formula* together with *blowing down* sequences in Section 4.

## 2 Some estimates and a monotonicity formula

**Lemma 2.1.** ([31, Lemma 2.2]) For any  $\zeta \in C^4(\mathbb{R}^N)$  and  $\eta \in C^4(\mathbb{R}^N)$ , the identity holds

$$\Delta\zeta\Delta(\zeta\eta^2) = [\Delta(\zeta\eta)]^2 - 4(\nabla\zeta \cdot \nabla\eta)^2 - \zeta^2|\Delta\eta|^2 + 2\zeta\Delta\zeta|\nabla\eta|^2 - 4\zeta\Delta\eta\nabla\zeta \cdot \nabla\eta.$$

**Lemma 2.2.** For any  $\zeta \in C^4(\mathbb{R}^N)$  and  $\eta \in C_0^4(\mathbb{R}^N)$ , then the following equalities hold

$$\begin{aligned} \int_{\mathbb{R}^N} \Delta \left( \frac{\Delta\zeta}{|x|^\beta} \right) \zeta \eta^2 dx &= \int_{\mathbb{R}^N} \frac{[\Delta(\zeta\eta)]^2}{|x|^\beta} + \int_{\mathbb{R}^N} \frac{1}{|x|^\beta} \left[ -4(\nabla\zeta \cdot \nabla\eta)^2 + 2\zeta\Delta\zeta|\nabla\eta|^2 \right] dx \\ &\quad + \int_{\mathbb{R}^N} \frac{\zeta^2}{|x|^\beta} \left[ 2\nabla(\Delta\eta) \cdot \nabla\eta + |\Delta\eta|^2 - 2\beta|x|^{-2}\Delta\eta(x \cdot \nabla\eta) \right] dx, \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} 2 \int_{\mathbb{R}^N} \frac{|\nabla\zeta|^2 |\nabla\eta|^2}{|x|^\beta} dx &= \int_{\mathbb{R}^N} \left[ \frac{2}{|x|^\beta} \zeta(-\Delta\zeta)|\nabla\eta|^2 + \frac{\zeta^2}{|x|^\beta} \Delta(|\nabla\eta|^2) \right] dx \\ &\quad + \int_{\mathbb{R}^N} \frac{\zeta^2}{|x|^{\beta+2}} \left[ \beta(\beta+2-N)|\nabla\eta|^2 - 2\beta(x \cdot \nabla(|\nabla\eta|^2)) \right] dx. \end{aligned} \quad (2.2)$$

*Proof.* By the divergence theorem and integration by parts, we get

$$\begin{aligned} -4 \int_{\mathbb{R}^N} \frac{1}{|x|^\beta} \zeta \Delta\eta \nabla\zeta \cdot \nabla\eta dx &= -2 \int_{\mathbb{R}^N} \frac{1}{|x|^\beta} \Delta\eta \nabla\zeta^2 \cdot \nabla\eta dx \\ &= 2 \int_{\mathbb{R}^N} \frac{\zeta^2}{|x|^\beta} \left[ \nabla(\Delta\eta) \cdot \nabla\eta + |\Delta\eta|^2 - \beta|x|^{-2}\Delta\eta(x \cdot \nabla\eta) \right] dx. \end{aligned}$$

Combining with Lemma 2.1, it implies that the identity (2.1) holds.

On the other hand, it is easy to see that

$$\frac{1}{2}\Delta(\zeta^2) = \zeta\Delta\zeta + |\nabla\zeta|^2,$$

then we obtain

$$\int_{\mathbb{R}^N} \frac{|\nabla\zeta|^2 |\nabla\eta|^2}{|x|^\beta} = \int_{\mathbb{R}^N} \frac{\zeta(-\Delta\zeta)|\nabla\eta|^2}{|x|^\beta} + \frac{1}{2} \int_{\mathbb{R}^N} \zeta^2 \Delta \left( \frac{|\nabla\eta|^2}{|x|^\beta} \right) dx.$$

A direct computation yields

$$\Delta \left( \frac{|\nabla\eta|^2}{|x|^\beta} \right) = \frac{1}{|x|^\beta} \left[ \beta(\beta+2-N)|x|^{-2}|\nabla\eta|^2 - 2\beta|x|^{-2}(x \cdot \nabla(|\nabla\eta|^2)) + \Delta(|\nabla\eta|^2) \right].$$

Substituting into the above identity, we get the identity (2.2).  $\square$

**Lemma 2.3.** *Let  $u \in C^4(\mathbb{R}^N)$  be a stable solution of (1.1). Then we find*

$$\begin{aligned} & \int_{B_R(x)} \left( \frac{|\Delta u|^2}{|z|^\beta} + |z|^\alpha |u|^{p+1} \right) dz \\ & \leq CR^{-2} \int_{B_{2R}(x) \setminus B_R(x)} \frac{|u \Delta u|}{|z|^\beta} dz + CR^{-4} \int_{B_{2R}(x) \setminus B_R(x)} \frac{u^2}{|z|^\beta} dz. \end{aligned} \quad (2.3)$$

Furthermore, for large enough  $m$ , we obtain that for any  $\psi \in C_0^4(\mathbb{R}^N)$  with  $0 \leq \psi \leq 1$

$$\begin{aligned} \int_{\mathbb{R}^N} \left[ \frac{|\Delta u|^2}{|x|^\beta} + |x|^\alpha |u|^{p+1} \right] \psi^{2m} dx & \leq C \int_{\mathbb{R}^N} |x|^{-\frac{2\alpha+\beta p+\beta}{p-1}} \mathfrak{Q}(\psi^m)^{\frac{p+1}{p-1}} dx \\ & + C \int_{\mathbb{R}^N} |x|^{-\frac{2\alpha+(\beta+2)(p+1)}{p-1}} \mathfrak{R}(\psi^m)^{\frac{p+1}{p-1}} dx, \end{aligned}$$

and

$$\int_{B_R(x)} \left[ \frac{|\Delta u|^2}{|z|^\beta} + |z|^\alpha |u|^{p+1} \right] \psi^{2m} dz \leq CR^{N-4-\beta-\frac{8+2\alpha+2\beta}{p-1}}. \quad (2.4)$$

Here

$$\begin{aligned} \mathfrak{Q}(\psi^m) &= |\nabla \psi|^4 + \psi^{2(2-m)} \left[ |\nabla(\Delta \psi^m) \cdot \nabla \psi^m| + |\Delta \psi^m|^2 + |\Delta |\nabla \psi^m|^2| \right], \\ \mathfrak{R}(\psi^m) &= \psi^{2(2-m)} \left[ |\Delta \psi^m| |x \cdot \nabla \psi^m| + |\nabla \psi^m|^2 + |x \cdot \nabla(|\nabla \psi^m|^2)| \right]. \end{aligned}$$

*proof.* From the definition of a stable solution  $u$ , it implies that if we take arbitrarily  $\zeta \in C_0^4(\mathbb{R}^N)$ , then we obtain

$$\int_{\mathbb{R}^N} |x|^\alpha |u|^{p-1} u \zeta dx = \int_{\mathbb{R}^N} \frac{\Delta u}{|x|^\beta} \Delta \zeta dx, \quad (2.5)$$

and

$$p \int_{\mathbb{R}^N} |x|^\alpha |u|^{p-1} \zeta^2 dx \leq \int_{\mathbb{R}^N} \frac{|\Delta \zeta|^2}{|x|^\beta} dx. \quad (2.6)$$

Now, in (2.5), we choose  $\zeta = u\psi^2$  with  $\psi \in C_0^4(\mathbb{R}^N)$ , and find

$$\int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} \psi^2 dx = \int_{\mathbb{R}^N} \frac{\Delta u}{|x|^\beta} \Delta(u\psi^2) dx. \quad (2.7)$$

We insert the test function  $\zeta = u\psi$  into (2.6) and get

$$p \int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} \psi^2 dx \leq \int_{\mathbb{R}^N} \frac{[\Delta(u\psi)]^2}{|x|^\beta} dx.$$

Putting the above inequality and (2.7) back into (2.1) yields

$$\begin{aligned} (p-1) \int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} \psi^2 dx & \leq \int_{\mathbb{R}^N} \frac{1}{|x|^\beta} \left[ 4(\nabla u \cdot \nabla \psi)^2 - 2u \Delta u |\nabla \psi|^2 \right] dx \\ & + \int_{\mathbb{R}^N} \frac{u^2}{|x|^\beta} \left[ 2|\nabla(\Delta \psi) \cdot \nabla \psi| + |\Delta \psi|^2 + 2\beta |x|^{-2} \Delta \psi (x \cdot \nabla \psi) \right] dx. \end{aligned}$$



Combining with the identity (2.2), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} \psi^2 dx &\leq C \int_{\mathbb{R}^N} \frac{|u \Delta u|}{|x|^\beta} |\nabla \psi|^2 dx \\
&+ C \int_{\mathbb{R}^N} \frac{u^2}{|x|^\beta} \left[ |\nabla(\Delta \psi) \cdot \nabla \psi| + |\Delta \psi|^2 + |\Delta |\nabla \psi|^2| \right] dx \\
&+ C \int_{\mathbb{R}^N} \frac{u^2}{|x|^{\beta+2}} \left[ |\Delta \psi| |x \cdot \nabla \psi| + |\nabla \psi|^2 + |x \cdot \nabla(|\nabla \psi|^2)| \right] dx. \quad (2.8)
\end{aligned}$$

Since  $\Delta(u\psi) = \Delta u \psi + 2\nabla u \cdot \nabla \psi + u \Delta \psi$ , it implies from (2.7), (2.8) and Lemma 2.2, that

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|\Delta u|^2}{|x|^\beta} \psi^2 dx &\leq C \int_{\mathbb{R}^N} \frac{|u \Delta u|}{|x|^\beta} |\nabla \psi|^2 dx + C \int_{\mathbb{R}^N} \frac{u^2}{|x|^\beta} \left[ |\nabla(\Delta \psi) \cdot \nabla \psi| + |\Delta \psi|^2 + |\Delta |\nabla \psi|^2| \right] dx \\
&+ C \int_{\mathbb{R}^N} \frac{u^2}{|x|^{\beta+2}} \left[ |\Delta \psi| |x \cdot \nabla \psi| + |\nabla \psi|^2 + |x \cdot \nabla(|\nabla \psi|^2)| \right] dx. \quad (2.9)
\end{aligned}$$

Replace  $\psi$  by  $\psi^m$  in (2.8) and (2.9) with  $m > 2$  to lead to

$$\begin{aligned}
\int_{\mathbb{R}^N} \left[ \frac{|\Delta u|^2}{|x|^\beta} + |x|^\alpha |u|^{p+1} \right] \psi^{2m} dx &\leq C \int_{\mathbb{R}^N} \frac{|u \Delta u|}{|x|^\beta} \psi^{2(m-1)} |\nabla \psi|^2 dx \\
&+ C \int_{\mathbb{R}^N} \frac{u^2}{|x|^\beta} \left[ |\nabla(\Delta \psi^m) \cdot \nabla \psi^m| + |\Delta \psi^m|^2 + |\Delta |\nabla \psi^m|^2| \right] dx \\
&+ C \int_{\mathbb{R}^N} \frac{u^2}{|x|^{\beta+2}} \left[ |\Delta \psi^m| |x \cdot \nabla \psi^m| + |\nabla \psi^m|^2 + |x \cdot \nabla(|\nabla \psi^m|^2)| \right] dx.
\end{aligned}$$

Utilizing Young's inequality, we obtain

$$\int_{\mathbb{R}^N} \frac{|u \Delta u|}{|x|^\beta} \psi^{2(m-1)} |\nabla \psi|^2 dx \leq \frac{1}{2C} \int_{\mathbb{R}^N} \frac{|\Delta u|^2}{|x|^\beta} \psi^{2m} dx + C \int_{\mathbb{R}^N} \frac{u^2}{|x|^\beta} \psi^{2(m-2)} |\nabla \psi|^4 dx.$$

Thus, it implies

$$\begin{aligned}
\int_{\mathbb{R}^N} \left[ \frac{|\Delta u|^2}{|x|^\beta} + |x|^\alpha |u|^{p+1} \right] \psi^{2m} dx &\leq C \int_{\mathbb{R}^N} \frac{u^2}{|x|^\beta} \psi^{2(m-2)} \mathfrak{Q}(\psi^m) dx \\
&+ C \int_{\mathbb{R}^N} \frac{u^2}{|x|^{\beta+2}} \psi^{2(m-2)} \mathfrak{R}(\psi^m) dx,
\end{aligned}$$

where  $\mathfrak{Q}(\psi^m) = |\nabla \psi|^4 + \psi^{2(2-m)} \left[ |\nabla(\Delta \psi^m) \cdot \nabla \psi^m| + |\Delta \psi^m|^2 + |\Delta |\nabla \psi^m|^2| \right]$  and  $\mathfrak{R}(\psi^m) = \psi^{2(2-m)} \left[ |\Delta \psi^m| |x \cdot \nabla \psi^m| + |\nabla \psi^m|^2 + |x \cdot \nabla(|\nabla \psi^m|^2)| \right]$ . Taking  $(m-2)(p+1) \geq 2m$ , we use Hölder's inequality to the both terms in the right hand side of the above inequality and get

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{u^2}{|x|^\beta} \psi^{2(m-2)} \mathfrak{Q}(\psi^m) dx &= \int_{\mathbb{R}^N} |x|^{\frac{2\alpha}{p+1}} u^2 \psi^{2(m-2)} |x|^{-\frac{2\alpha}{p+1}-\beta} \mathfrak{Q}(\psi^m) dx \\
&\leq \left( \int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} \psi^{2m} dx \right)^{\frac{2}{p+1}} \left( \int_{\mathbb{R}^N} |x|^{-\frac{2\alpha+\beta p+\beta}{p-1}} \mathfrak{Q}(\psi^m)^{\frac{p+1}{p-1}} dx \right)^{\frac{p-1}{p+1}},
\end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{\beta+2}} \psi^{2(m-2)} \Re(\psi^m) dx &= \int_{\mathbb{R}^N} |x|^{\frac{2\alpha}{p+1}} u^2 \psi^{2(m-2)} |x|^{-\frac{2\alpha}{p+1}-\beta-2} \Re(\psi^m) dx \\ &\leq \left( \int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} \psi^{2m} dx \right)^{\frac{2}{p+1}} \left( \int_{\mathbb{R}^N} |x|^{-\frac{2\alpha+(\beta+2)(p+1)}{p-1}} \Re(\psi^m)^{\frac{p+1}{p-1}} dx \right)^{\frac{p-1}{p+1}}. \end{aligned}$$

Therefore, we find

$$\begin{aligned} \int_{\mathbb{R}^N} \left[ \frac{|\Delta u|^2}{|x|^\beta} + |x|^\alpha |u|^{p+1} \right] \psi^{2m} dx &\leq C \int_{\mathbb{R}^N} |x|^{-\frac{2\alpha+\beta p+\beta}{p-1}} \Re(\psi^m)^{\frac{p+1}{p-1}} dx \\ &\quad + C \int_{\mathbb{R}^N} |x|^{-\frac{2\alpha+(\beta+2)(p+1)}{p-1}} \Re(\psi^m)^{\frac{p+1}{p-1}} dx. \end{aligned}$$

Let us choose  $\psi \in C_0^4(B_{2R}(x))$  a cut-off function verifying  $0 \leq \psi \leq 1$ ,  $\psi \equiv 1$  in  $B_R(x)$ , and  $|\nabla^k \psi| \leq \frac{C}{R^k}$  for  $k \leq 3$ . Substituting  $\psi$  into (2.8), (2.9) and the above inequality, we have

$$\begin{aligned} &\int_{B_R(x)} \left( \frac{|\Delta u|^2}{|z|^\beta} + |z|^\alpha |u|^{p+1} \right) dz \\ &\leq CR^{-2} \int_{B_{2R}(x) \setminus B_R(x)} \frac{|u \Delta u|}{|z|^\beta} dz + CR^{-4} \int_{B_{2R}(x) \setminus B_R(x)} \frac{u^2}{|z|^\beta} dz. \end{aligned}$$

and

$$\int_{B_R(x)} \left[ \frac{|\Delta u|^2}{|z|^\beta} + |z|^\alpha |u|^{p+1} \right] \psi^{2m} dz \leq CR^{N-4-\beta-\frac{8+2\alpha+2\beta}{p-1}}. \quad \square$$

**Remark 2.1.** If the domain  $\mathbb{R}^N$  is replaced by the subset  $\Omega$  (boundedness or not) in Lemma 2.1-Lemma 2.3, then the conclusions are also true.

**Lemma 2.4.** (Pohozaev identity) Let  $u$  be a classical solution of (1.1), then we have

$$\begin{aligned} &\frac{N-4-\beta}{2} \int_{\Omega} \frac{|\Delta u|^2}{|x|^\beta} dx - \frac{N+\alpha}{p+1} \int_{\Omega} |x|^\alpha |u|^{p+1} dx \\ &= \frac{1}{2} \int_{\partial\Omega} \frac{|\Delta u|^2}{|x|^\beta} (x \cdot \nu) dS - \frac{1}{p+1} \int_{\partial\Omega} |x|^\alpha |u|^{p+1} (x \cdot \nu) dS \\ &\quad - \int_{\partial\Omega} \frac{\Delta u}{|x|^\beta} \nabla(x \cdot \nabla u) \cdot \nu dS + \int_{\partial\Omega} \nabla \left( \frac{\Delta u}{|x|^\beta} \right) \cdot \nu (x \cdot \nabla u) dS, \end{aligned} \quad (2.10)$$

where  $\nu$  denotes the outward unit normal vector field.

*Proof.* Multiplying (1.1) by  $(x \cdot \nabla u)$ , we obtain

$$\Delta(|x|^{-\beta} \Delta u)(x \cdot \nabla u) = |x|^\alpha |u|^{p-1} u (x \cdot \nabla u), \quad \text{in } \Omega \setminus \{0\}.$$

Hence, for every small  $\varepsilon > 0$ , we have

$$\int_{\Omega \setminus B_\varepsilon} \Delta(|x|^{-\beta} \Delta u)(x \cdot \nabla u) dx = \int_{\Omega \setminus B_\varepsilon} |x|^\alpha |u|^{p-1} u (x \cdot \nabla u) dx. \quad (2.11)$$

Apply the divergence theorem and integration by parts to calculate the right hand side and the left hand side of (2.11) respectively, and get

$$\begin{aligned} \frac{1}{p+1} \int_{\Omega \setminus B_\varepsilon} |x|^\alpha (x \cdot \nabla (|u|^{p+1})) dx &= -\frac{N+\alpha}{p+1} \int_{\Omega \setminus B_\varepsilon} |x|^\alpha |u|^{p+1} dx \\ &+ \frac{1}{p+1} \int_{\partial\Omega} |x|^\alpha |u|^{p+1} (x \cdot \nu) dS - \frac{1}{p+1} \int_{\partial B_\varepsilon} |x|^\alpha |u|^{p+1} (x \cdot \nu) dS, \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} \int_{\Omega \setminus B_\varepsilon} \Delta(|x|^{-\beta} \Delta u) (x \cdot \nabla u) dx &= \sum_{i,j=1}^N \int_{\Omega \setminus B_\varepsilon} (|x|^{-\beta} \Delta u)_{x_i x_j} (x^j u_{x_i}) dx \\ &= \sum_{i,j=1}^N \int_{\Omega \setminus B_\varepsilon} (|x|^{-\beta} \Delta u) (x^j u_{x_j})_{x_i x_i} dx - \int_{\partial\Omega} |x|^{-\beta} \Delta u \nabla(x \cdot \nabla u) \cdot \nu dS \\ &\quad + \int_{\partial B_\varepsilon} |x|^{-\beta} \Delta u \nabla(x \cdot \nabla u) \cdot \nu dS + \int_{\partial\Omega} \nabla(|x|^{-\beta} \Delta u) \cdot \nu (x \cdot \nabla u) dS \\ &\quad - \int_{\partial B_\varepsilon} \nabla(|x|^{-\beta} \Delta u) \cdot \nu (x \cdot \nabla u) dS. \end{aligned}$$

Again computing the first term in the right hand side of the above equality yields

$$\begin{aligned} \sum_{i,j=1}^N \int_{\Omega \setminus B_\varepsilon} (|x|^{-\beta} \Delta u) (x^j u_{x_j})_{x_i x_i} dx &= \sum_{i,j=1}^N \int_{\Omega \setminus B_\varepsilon} |x|^{-\beta} \Delta u \left[ 2\delta^{ij} u_{x_i x_j} + x^j u_{x_i x_i x_j} \right] dx \\ &= -\frac{N-4-\beta}{2} \int_{\Omega \setminus B_\varepsilon} |x|^{-\beta} |\Delta u|^2 dx + \int_{\partial\Omega} \frac{|\Delta u|^2}{2} |x|^{-\beta} x \cdot \nu dS - \int_{\partial B_\varepsilon} \frac{|\Delta u|^2}{2} |x|^{-\beta} x \cdot \nu dS, \end{aligned}$$

and putting back into the above equality leads to

$$\begin{aligned} \int_{\Omega \setminus B_\varepsilon} \Delta(|x|^{-\beta} \Delta u) (x \cdot \nabla u) dx &= -\frac{N-4-\beta}{2} \int_{\Omega \setminus B_\varepsilon} \frac{|\Delta u|^2}{|x|^\beta} dx + \int_{\partial\Omega} \frac{|\Delta u|^2}{2} |x|^{-\beta} x \cdot \nu dS \\ &\quad - \int_{\partial\Omega} \frac{\Delta u}{|x|^\beta} \nabla(x \cdot \nabla u) \cdot \nu dS + \int_{\partial\Omega} \nabla \left( \frac{\Delta u}{|x|^\beta} \right) \cdot \nu (x \cdot \nabla u) dS - \int_{\partial B_\varepsilon} \frac{|\Delta u|^2}{2} |x|^{-\beta} x \cdot \nu dS \\ &\quad + \int_{\partial B_\varepsilon} \frac{\Delta u}{|x|^\beta} \nabla(x \cdot \nabla u) \cdot \nu dS - \int_{\partial B_\varepsilon} \nabla \left( \frac{\Delta u}{|x|^\beta} \right) \cdot \nu (x \cdot \nabla u) dS. \end{aligned} \quad (2.13)$$

Since  $u \in C^4(\Omega)$ ,  $\alpha > -4$  and  $0 \leq \beta \leq \frac{N-4}{2}$ , the above integrations are well-defined. Now, we insert (2.12) and (2.13) into (2.11), take  $\varepsilon \rightarrow 0$  and pass to the limit to obtain the identity (2.10).  $\square$

Inspired by the ideas of [9, 10, 17], we will apply *Pohozaev identity* to construct a monotonicity formula which is a crucial tool. More precisely, choose  $u \in W_{loc}^{4,2}(\Omega)$  and

$|x|^\alpha |u|^{p+1} \in L^1_{loc}(\Omega)$ , fix  $x \in \Omega$ , let  $0 < r < R$  and  $B_r(x) \subset B_R(x) \subset \Omega$ , and define

$$\begin{aligned}
\mathcal{M}(r; x, u) = & r^\delta \int_{B_r(x)} \frac{1}{2} \frac{|\Delta u|^2}{|z|^\beta} - \frac{1}{p+1} |z|^\alpha |u|^{p+1} \\
& + \frac{(1+\beta)\lambda}{2} (N-2-\lambda) \left( r^{2\lambda+1-N} \int_{\partial B_r(x)} u^2 \right) \\
& + \frac{\lambda}{2} (N-2-\lambda) \frac{d}{dr} \left( r^{2\lambda+2-N} \int_{\partial B_r(x)} u^2 \right) \\
& + \frac{r^3}{2} \frac{d}{dr} \left[ r^{2\lambda+1-N} \int_{\partial B_r(x)} \left( \lambda r^{-1} u + \frac{\partial u}{\partial r} \right)^2 \right] \\
& + \frac{1+\beta-\lambda}{2} r^{2\lambda+3-N} \int_{\partial B_r(x)} \left( |\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right) \\
& + \frac{1}{2} \frac{d}{dr} \left[ r^{2\lambda+4-N} \int_{\partial B_r(x)} \left( |\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right) \right]. \tag{2.14}
\end{aligned}$$

Here and in the following, we always set  $\delta := \frac{8+2\alpha+2\beta}{p-1} + 4 + \beta - N$  and  $\lambda := \frac{4+\alpha+\beta}{p-1}$ .

**Theorem 2.1.** *Let  $p \geq \frac{N+4+2\alpha+\beta}{N-4-\beta}$  and let  $u \in W^{2,2}_{loc}(\Omega)$ ,  $|x|^\alpha |u|^{p+1} \in L^1_{loc}(\Omega)$  and  $|x|^{-\beta} |\Delta u|^2 \in L^1_{loc}(\Omega)$  be a weak solution of (1.1). Then  $\mathcal{M}(r; x, u)$  is nondecreasing in  $r \in (0, R)$  and satisfies the inequality*

$$\frac{d}{dr} \mathcal{M}(r; 0, u) \geq C(N, p, \alpha, \beta) r^{-N+2+2\lambda} \int_{\partial B_r} \left( \lambda r^{-1} u + \frac{\partial u}{\partial r} \right)^2 dS,$$

where  $C(N, p, \alpha, \beta) = (N-2)(2+\beta) + 2\lambda(N-4-\beta-\lambda) - \frac{\beta^2}{8} > 0$ .

Furthermore, if  $\mathcal{M}(r; 0, u) \equiv \text{const}$ , for all  $r \in (0, R)$ , then  $u$  is homogeneous in  $B_R \setminus \{0\}$ , i.e.,  $\forall \mu \in (0, 1]$ ,  $x \in B_R \setminus \{0\}$ ,

$$u(\mu x) = \mu^{-\frac{4+\alpha+\beta}{p-1}} u(x).$$

*Proof.* Define a function by

$$\mathcal{F}(\kappa) := \kappa^\delta \int_{B_\kappa} \left( \frac{1}{2} \frac{|\Delta u|^2}{|x|^\beta} - \frac{1}{p+1} |x|^\alpha |u|^{p+1} \right) dx.$$

Differentiating the function  $\mathcal{F}(\kappa)$  in  $\kappa$  arrives at

$$\begin{aligned}
\frac{d\mathcal{F}(\kappa)}{d\kappa} = & \delta \kappa^{\delta-1} \int_{B_\kappa} \left( \frac{1}{2} \frac{|\Delta u|^2}{|x|^\beta} - \frac{1}{p+1} |x|^\alpha |u|^{p+1} \right) dx \\
& + \kappa^\delta \int_{\partial B_\kappa} \left( \frac{1}{2} \frac{|\Delta u|^2}{|x|^\beta} - \frac{1}{p+1} |x|^\alpha |u|^{p+1} \right) dS. \tag{2.15}
\end{aligned}$$

Multiply the equation (1.1) by  $u$  and integrate by parts to get

$$\begin{aligned} \int_{\Omega} |x|^{\alpha} |u|^{p+1} dx &= - \int_{\Omega} \nabla \left( \frac{\Delta u}{|x|^{\beta}} \right) \cdot \nabla u dx + \int_{\partial\Omega} \frac{\partial}{\partial \nu} \left( \frac{\Delta u}{|x|^{\beta}} \right) u dS \\ &= \int_{\Omega} \frac{|\Delta u|^2}{|x|^{\beta}} dx - \int_{\partial\Omega} \frac{\Delta u}{|x|^{\beta}} (\nabla u \cdot \nu) dS + \int_{\partial\Omega} \frac{\partial}{\partial \nu} \left( \frac{\Delta u}{|x|^{\beta}} \right) u dS, \end{aligned}$$

implies

$$\int_{\Omega} \frac{|\Delta u|^2}{|x|^{\beta}} dx - \int_{\Omega} |x|^{\alpha} |u|^{p+1} dx = \int_{\partial\Omega} \frac{\Delta u}{|x|^{\beta}} (\nabla u \cdot \nu) dS - \int_{\partial\Omega} \frac{\partial}{\partial \nu} \left( \frac{\Delta u}{|x|^{\beta}} \right) u dS. \quad (2.16)$$

In addition, it is easily to see that

$$\begin{aligned} &\delta \kappa^{\delta-1} \int_{B_{\kappa}} \left( \frac{1}{2} \frac{|\Delta u|^2}{|x|^{\beta}} - \frac{1}{p+1} |x|^{\alpha} |u|^{p+1} \right) dx \\ &= \lambda \kappa^{\delta-1} \int_{B_{\kappa}} \left( \frac{|\Delta u|^2}{|x|^{\beta}} - |x|^{\alpha} |u|^{p+1} \right) dx - \kappa^{\delta-1} \int_{B_{\kappa}} \left( \frac{N-4-\beta}{2} \frac{|\Delta u|^2}{|x|^{\beta}} - \frac{N+\alpha}{p+1} |x|^{\alpha} |u|^{p+1} \right) dx. \end{aligned}$$

Therefore, combining the result with (2.10), (2.15) and (2.16), we obtain

$$\begin{aligned} \frac{d\mathcal{F}(\kappa)}{d\kappa} &= \lambda \kappa^{\delta-1} \left[ \int_{\partial B_{\kappa}} \frac{\Delta u}{|x|^{\beta}} (\nabla u \cdot \nu) dS - \int_{\partial B_{\kappa}} \frac{\partial}{\partial \nu} \left( \frac{\Delta u}{|x|^{\beta}} \right) u dS \right] \\ &\quad + \kappa^{\delta-1} \int_{\partial B_{\kappa}} \frac{\Delta u}{|x|^{\beta}} \nabla(x \cdot \nabla u) \cdot \nu dS - \kappa^{\delta-1} \int_{\partial B_{\kappa}} \nabla \left( \frac{\Delta u}{|x|^{\beta}} \right) \cdot \nu (x \cdot \nabla u) dS. \end{aligned} \quad (2.17)$$

Denote  $u^{\kappa}(x) := \kappa^{\frac{4+\alpha+\beta}{p-1}} u(\kappa x)$ . Now, computing the first term in the right hand side of (2.17) leads to

$$\begin{aligned} \lambda \kappa^{\delta-1} \int_{\partial B_{\kappa}} |x|^{-\beta} \Delta u (\nabla u \cdot \nu) dS &= \lambda \kappa^2 \int_{\partial B_R} \Delta \kappa^{\frac{4+\alpha+\beta}{p-1}} u \left( \nabla \kappa^{\frac{4+\alpha+\beta}{p-1}} u \cdot \nu \right) dS \cdot \kappa^{1-N} \\ &= \frac{\lambda}{\kappa} \int_{\partial B_1} \Delta u^{\kappa} (\nabla u^{\kappa} \cdot \nu) d\sigma. \end{aligned} \quad (2.18)$$

Similarly, we calculate the second term in the right hand side of (2.17) and get

$$\begin{aligned} \lambda \kappa^{\delta-1} \int_{\partial B_{\kappa}} \frac{\partial}{\partial \nu} \left( \frac{\Delta u}{|x|^{\beta}} \right) u dS &= \lambda \kappa^{\delta-1} \int_{\partial B_{\kappa}} |x|^{-\beta} [(\nabla(\Delta u) \cdot \nu) - \beta |x|^{-2} \Delta u (x \cdot \nu)] u dS \\ &= \lambda \kappa^2 \int_{\partial B_{\kappa}} \left[ (\nabla \left( \Delta \kappa^{\frac{4+\alpha+\beta}{p-1}} u \right) \cdot \nu) - \beta \kappa^{-2} \Delta \left( \kappa^{\frac{4+\alpha+\beta}{p-1}} u \right) (x \cdot \nu) \right] \kappa^{\frac{4+\alpha+\beta}{p-1}} u dS \cdot \kappa^{1-N} \\ &= \frac{\lambda}{\kappa} \int_{\partial B_1} [(\nabla(\Delta u^{\kappa}) \cdot \nu) - \beta \Delta u^{\kappa}] u^{\kappa} d\sigma. \end{aligned} \quad (2.19)$$

Similar to the above calculation, we find

$$\kappa^{\delta-1} \int_{\partial B_{\kappa}} |x|^{-\beta} \Delta u \nabla(x \cdot \nabla u) \cdot \nu dS = \frac{1}{\kappa} \int_{\partial B_1} \Delta u^{\kappa} \nabla(x \cdot \nabla u^{\kappa}) \cdot \nu d\sigma, \quad (2.20)$$

and

$$\kappa^{\delta-1} \int_{\partial B_\kappa} \nabla(|x|^{-\beta} \Delta u) \cdot \nu (x \cdot \nabla u) dS = \frac{1}{\kappa} \int_{\partial B_1} \left[ (\nabla(\Delta u^\kappa) \cdot \nu) - \beta \Delta u^\kappa \right] (x \cdot \nabla u^\kappa) d\sigma. \quad (2.21)$$

We use spherical coordinates  $r = |x|$ ,  $\theta = \frac{x}{|x|} \in \mathbb{S}^{N-1}$  and write  $u^\kappa(x) = u^\kappa(r, \theta)$ , then we insert (2.18)-(2.21) into (2.17) to obtain

$$\begin{aligned} \frac{d\mathcal{F}(\kappa)}{d\kappa} &= \frac{\lambda}{\kappa} \int_{\partial B_1} \Delta u^\kappa (\nabla u^\kappa \cdot \nu) d\sigma - \frac{\lambda}{\kappa} \int_{\partial B_1} \left[ (\nabla(\Delta u^\kappa) \cdot \nu) - \beta \Delta u^\kappa \right] u^\kappa d\sigma \\ &\quad + \frac{1}{\kappa} \int_{\partial B_1} \Delta u^\kappa \nabla (x \cdot \nabla u^\kappa) \cdot \nu d\sigma - \frac{1}{\kappa} \int_{\partial B_1} \left[ (\nabla(\Delta u^\kappa) \cdot \nu) - \beta \Delta u^\kappa \right] (x \cdot \nabla u^\kappa) d\sigma \\ &= \frac{1}{\kappa} \int_{\partial B_1} \lambda \left( \frac{\partial^2 u^\kappa}{\partial r^2} + (N-1) \frac{\partial u^\kappa}{\partial r} + \Delta_\theta u^\kappa \right) \frac{\partial u^\kappa}{\partial r} \\ &\quad - \lambda \left[ \frac{\partial^3 u^\kappa}{\partial r^3} + (N-1-\beta) \frac{\partial^2 u^\kappa}{\partial r^2} - (N-1)(1+\beta) \frac{\partial u^\kappa}{\partial r} - (2+\beta) \Delta_\theta u^\kappa \right] u^\kappa \\ &\quad + \left[ \frac{\partial^2 u^\kappa}{\partial r^2} + (N-1) \frac{\partial u^\kappa}{\partial r} + \Delta_\theta u^\kappa \right] \left( \frac{\partial^2 u^\kappa}{\partial r^2} + \frac{\partial u^\kappa}{\partial r} \right) \\ &\quad - \left[ \frac{\partial^3 u^\kappa}{\partial r^3} + (N-1-\beta) \frac{\partial^2 u^\kappa}{\partial r^2} - (N-1)(1+\beta) \frac{\partial u^\kappa}{\partial r} - (2+\beta) \Delta_\theta u^\kappa \right] \frac{\partial u^\kappa}{\partial r} \\ &= \frac{1}{\kappa} \int_{\partial B_1} -\frac{\partial^3 u^\kappa}{\partial r^3} \frac{\partial u^\kappa}{\partial r} - \lambda \frac{\partial^3 u^\kappa}{\partial r^3} u^\kappa + \left( \frac{\partial^2 u^\kappa}{\partial r^2} \right)^2 + (\lambda+1+\beta) \frac{\partial^2 u^\kappa}{\partial r^2} \frac{\partial u^\kappa}{\partial r} \\ &\quad + (N-1)(\lambda+2+\beta) \left( \frac{\partial u^\kappa}{\partial r} \right)^2 - \lambda(N-1-\beta) \frac{\partial^2 u^\kappa}{\partial r^2} u^\kappa + \lambda(N-1)(1+\beta) \frac{\partial u^\kappa}{\partial r} u^\kappa \\ &\quad + \frac{1}{\kappa} \int_{\partial B_1} \Delta_\theta u^\kappa \frac{\partial^2 u^\kappa}{\partial r^2} + (\lambda+3+\beta) \Delta_\theta u^\kappa \frac{\partial u^\kappa}{\partial r} + \lambda(2+\beta) \Delta_\theta u^\kappa u^\kappa \\ &:= \mathfrak{T}_1 + \mathfrak{T}_2, \end{aligned}$$

where  $\Delta_\theta$  represents the Laplace-Beltrami operator on  $\partial B_1$  and  $\nabla_\theta$  is the tangential derivative on  $\partial B_1$ .

Differentiating  $u^\kappa$  in  $\kappa$  implies

$$\frac{du^\kappa}{d\kappa}(x) = \frac{1}{\kappa} \left[ \lambda u^\kappa(x) + r \frac{\partial u^\kappa}{\partial r}(x) \right] \implies r \frac{\partial u^\kappa}{\partial r} = \kappa \frac{du^\kappa}{d\kappa} - \lambda u^\kappa. \quad (2.22)$$

Differentiating the equation (2.22) in  $\kappa$  and  $r$  respectively yields

$$r \frac{\partial}{\partial r} \frac{du^\kappa}{d\kappa} = \kappa \frac{d^2 u^\kappa}{d\kappa^2} + (1-\lambda) \frac{du^\kappa}{d\kappa} \quad \text{and} \quad \kappa \frac{\partial}{\partial r} \frac{du^\kappa}{d\kappa} = (1+\lambda) \frac{\partial u^\kappa}{\partial r} + r \frac{\partial^2 u^\kappa}{\partial r^2}.$$

Then, combining the above two equalities with (2.22), we obtain that, on  $\partial B_1$

$$\frac{\partial^2 u^\kappa}{\partial r^2} = r^2 \frac{\partial^2 u^\kappa}{\partial r^2} = \kappa^2 \frac{d^2 u^\kappa}{d\kappa^2} - 2\lambda \kappa \frac{du^\kappa}{d\kappa} + \lambda(1+\lambda) u^\kappa. \quad (2.23)$$

Similarly, we find

$$\begin{aligned} r^2 \frac{\partial^3 u}{\partial r^3} + 2r \frac{\partial^2 u}{\partial r^2} &= \kappa^2 \frac{\partial}{\partial r} \frac{d^2 u^\kappa}{d\kappa^2} - 2\lambda\kappa \frac{\partial}{\partial r} \frac{du^\kappa}{d\kappa} + \lambda(1+\lambda) \frac{\partial u^\kappa}{\partial r}, \\ r \frac{\partial}{\partial r} \frac{d^2 u^\kappa}{d\kappa^2} &= \kappa \frac{d^3 u^\kappa}{d\kappa^3} + (2-\lambda) \frac{d^2 u^\kappa}{d\kappa^2}. \end{aligned}$$

Then, on  $\partial B_1$ , we have

$$\frac{\partial^3 u}{\partial r^3} = \kappa^3 \frac{d^3 u^\kappa}{d\kappa^3} - 3\lambda\kappa^2 \frac{d^2 u^\kappa}{d\kappa^2} + 3\lambda(1+\lambda)\kappa \frac{du^\kappa}{d\kappa} - \lambda(1+\lambda)(2+\lambda)u^\kappa. \quad (2.24)$$

Substituting (2.22) and (2.23) into the expression of  $\mathfrak{T}_2$  arrives at

$$\begin{aligned} \mathfrak{T}_2 &= \int_{\partial B_1} \kappa \Delta_\theta u^\kappa \frac{d^2 u^\kappa}{d\kappa^2} - (\lambda - 3 - \beta) \Delta_\theta u^\kappa \frac{du^\kappa}{d\kappa} \\ &= \int_{\partial B_1} -\kappa \nabla_\theta u^\kappa \nabla_\theta \frac{d^2 u^\kappa}{d\kappa^2} + (\lambda - 3 - \beta) \nabla_\theta u^\kappa \nabla_\theta \frac{du^\kappa}{d\kappa} \\ &= -\frac{1}{2} \frac{d^2}{d\kappa^2} \left[ \kappa \int_{\partial B_1} |\nabla_\theta u^\kappa|^2 \right] + \frac{\lambda - 1 - \beta}{2} \frac{d}{d\kappa} \int_{\partial B_1} |\nabla_\theta u^\kappa|^2 + \kappa \int_{\partial B_1} \left| \nabla_\theta \frac{du^\kappa}{d\kappa} \right|^2 \\ &\geq -\frac{1}{2} \frac{d^2}{d\kappa^2} \left[ \kappa \int_{\partial B_1} |\nabla_\theta u^\kappa|^2 \right] + \frac{\lambda - 1 - \beta}{2} \frac{d}{d\kappa} \int_{\partial B_1} |\nabla_\theta u^\kappa|^2. \end{aligned}$$

Let us note the two equalities

$$\begin{aligned} -\kappa^3 \frac{du^\kappa}{d\kappa} \frac{d^3 u^\kappa}{d\kappa^3} &= \frac{d}{d\kappa} \left( -\frac{\kappa^3}{2} \frac{d}{d\kappa} \left( \frac{du^\kappa}{d\kappa} \right)^2 \right) + 3\kappa^2 \frac{du^\kappa}{d\kappa} \frac{d^2 u^\kappa}{d\kappa^2} + \kappa^3 \left( \frac{d^2 u^\kappa}{d\kappa^2} \right)^2, \\ \kappa u^\kappa \frac{d^2 u^\kappa}{d\kappa^2} &= \frac{d^2}{d\kappa^2} \left( \frac{\kappa (u^\kappa)^2}{2} \right) - 2u^\kappa \frac{du^\kappa}{d\kappa} - \kappa \left( \frac{du^\kappa}{d\kappa} \right)^2. \end{aligned}$$

Inserting (2.22)-(2.24) into the expression of  $\mathfrak{T}_1$ , and combining with the above two equalities, we get

$$\begin{aligned} \mathfrak{T}_1 &= \int_{\partial B_1} -\kappa^3 \frac{d^3 u^\kappa}{d\kappa^3} \frac{du^\kappa}{d\kappa} + \kappa^3 \left( \frac{d^2 u^\kappa}{d\kappa^2} \right)^2 + (1+\beta)\kappa^2 \frac{d^2 u^\kappa}{d\kappa^2} \frac{du^\kappa}{d\kappa} \\ &\quad + \left[ (N-1)(2+\lambda+\beta) - \lambda(5+\lambda+2\beta) \right] \kappa \left( \frac{du^\kappa}{d\kappa} \right)^2 \\ &\quad + \lambda(2+\lambda-N)\kappa u^\kappa \frac{d^2 u^\kappa}{d\kappa^2} + \lambda(3+\beta)(\lambda+2-N)u^\kappa \frac{du^\kappa}{d\kappa} \\ &= \int_{\partial B_1} \frac{d}{d\kappa} \left( -\frac{\kappa^3}{2} \frac{d}{d\kappa} \left( \frac{du^\kappa}{d\kappa} \right)^2 \right) + 2\kappa^3 \left( \frac{d^2 u^\kappa}{d\kappa^2} \right)^2 + (4+\beta)\kappa^2 \frac{d^2 u^\kappa}{d\kappa^2} \frac{du^\kappa}{d\kappa} \\ &\quad + \left[ (N-1)(2+\beta) + 2\lambda(N-4-\beta-\lambda) \right] \kappa \left( \frac{du^\kappa}{d\kappa} \right)^2 \\ &\quad + \lambda(2+\lambda-N) \frac{d^2}{d\kappa^2} \left( \frac{\kappa (u^\kappa)^2}{2} \right) + \lambda(2+\lambda-N)(1+\beta)u^\kappa \frac{du^\kappa}{d\kappa} \end{aligned}$$

$$\begin{aligned} &\geq \int_{\partial B_1} \frac{d}{d\kappa} \left( -\frac{\kappa^3}{2} \frac{d}{d\kappa} \left( \frac{du^\kappa}{d\kappa} \right)^2 \right) + \frac{\lambda(2+\lambda-N)}{2} \frac{d^2}{d\kappa^2} (\kappa(u^\kappa)^2) \\ &\quad + \frac{\lambda}{2} (2+\lambda-N)(1+\beta) \frac{d}{d\kappa} (u^\kappa)^2. \end{aligned}$$

Since  $p \geq \frac{N+4+2\alpha+\beta}{N-4-\beta}$ , the deleted terms of  $\mathfrak{T}_1$  satisfies

$$\begin{aligned} &2\kappa^3 \left( \frac{d^2 u^\kappa}{d\kappa^2} \right)^2 + (4+\beta)\kappa^2 \frac{d^2 u^\kappa}{d\kappa^2} \frac{du^\kappa}{d\kappa} + \left[ (N-1)(2+\beta) + 2\lambda(N-4-\beta-\lambda) \right] \kappa \left( \frac{du^\kappa}{d\kappa} \right)^2 \\ &= 2\kappa \left[ \kappa \frac{d^2 u^\kappa}{d\kappa^2} + \left( 1 + \frac{\beta}{4} \right) \frac{du^\kappa}{d\kappa} \right]^2 + C(N, p, \alpha, \beta) \kappa \left( \frac{du^\kappa}{d\kappa} \right)^2 \geq 0, \end{aligned}$$

where

$$\begin{aligned} C(N, p, \alpha, \beta) &= (N-1)(2+\beta) + 2\lambda(N-4-\beta-\lambda) - 2 \left( 1 + \frac{\beta}{4} \right)^2 \\ &= (N-2)(2+\beta) + 2\lambda(N-4-\beta-\lambda) - \frac{\beta^2}{8} > 0. \end{aligned}$$

Now, we rescale and write those  $\kappa$  derivatives in  $\mathfrak{T}_1$  and  $\mathfrak{T}_2$  as follows.

$$\begin{aligned} \int_{\partial B_1} \frac{d}{d\kappa} (u^\kappa)^2 &= \frac{d}{d\kappa} \left( \kappa^{2\lambda+1-N} \int_{\partial B_\kappa} u^2 \right), \\ \int_{\partial B_1} \frac{d^2}{d\kappa^2} [\kappa (u^\kappa)^2] &= \frac{d^2}{d\kappa^2} \left( \kappa^{2\lambda+2-N} \int_{\partial B_\kappa} u^2 \right), \\ \int_{\partial B_1} \frac{d}{d\kappa} \left[ \kappa^3 \frac{d}{d\kappa} \left( \frac{du^\kappa}{d\kappa} \right)^2 \right] &= \frac{d}{d\kappa} \left[ \kappa^3 \frac{d}{d\kappa} \left( \kappa^{2\lambda+1-N} \int_{\partial B_\kappa} \left( \lambda \kappa^{-1} u + \frac{\partial u}{\partial r} \right)^2 \right) \right], \\ \frac{d}{d\kappa} \left( \int_{\partial B_1} |\nabla_\theta u^\kappa|^2 \right) &= \frac{d}{d\kappa} \left[ \kappa^{2\lambda+3-N} \int_{\partial B_\kappa} \left( |\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right) \right], \\ \frac{d^2}{d\kappa^2} \left( \kappa \int_{\partial B_1} |\nabla_\theta u^\kappa|^2 \right) &= \frac{d^2}{d\kappa^2} \left[ \kappa^{2\lambda+4-N} \int_{\partial B_\kappa} \left( |\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right) \right]. \end{aligned}$$

Substituting these terms into  $\frac{d\mathcal{F}(\kappa)}{d\kappa}$  yields

$$\begin{aligned} \frac{d\mathcal{F}(\kappa)}{d\kappa} &\geq \frac{\lambda(2+\lambda-N)(1+\beta)}{2} \frac{d}{d\kappa} \left( \kappa^{2\lambda+1-N} \int_{\partial B_\kappa} u^2 \right) \\ &\quad + \frac{\lambda(2+\lambda-N)}{2} \frac{d^2}{d\kappa^2} \left( \kappa^{2\lambda+2-N} \int_{\partial B_\kappa} u^2 \right) \\ &\quad - \frac{1}{2} \frac{d}{d\kappa} \left[ \kappa^3 \frac{d}{d\kappa} \left( \kappa^{2\lambda+1-N} \int_{\partial B_\kappa} \left( \lambda \kappa^{-1} u + \frac{\partial u}{\partial r} \right)^2 \right) \right] \\ &\quad + \frac{\lambda-1-\beta}{2} \frac{d}{d\kappa} \left[ \kappa^{2\lambda+3-N} \int_{\partial B_\kappa} \left( |\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right) \right] \\ &\quad - \frac{1}{2} \frac{d^2}{d\kappa^2} \left[ \kappa^{2\lambda+4-N} \int_{\partial B_\kappa} \left( |\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right) \right]. \end{aligned}$$



Applying the properties of integration, we conclude that  $\mathcal{M}(r; x, u)$  is well defined and non-decreasing in  $r \in (0, R)$ .

Next, we let  $\mathcal{M}(r; 0, u) \equiv \text{const}$ , for all  $r \in (0, R)$ , then, for any  $r_1, r_2 \in (0, R)$  with  $r_1 < r_2$ , we have

$$\begin{aligned} 0 &= \mathcal{M}(r_2; 0, u) - \mathcal{M}(r_1; 0, u) = \int_{r_1}^{r_2} \frac{d}{d\mu} \mathcal{M}(\mu; 0, u) d\mu \\ &\geq C(N, p, \alpha, \beta) \int_{B_{r_2} \setminus B_{r_1}} |x|^{2+2\lambda-N} \left( \lambda\mu^{-1}u + \frac{\partial u}{\partial \mu} \right)^2 dx. \end{aligned}$$

Thus, we get

$$\lambda\mu^{-1}u + \frac{\partial u}{\partial \mu} = 0, \quad \text{a.e. in } B_R \setminus \{0\}.$$

Integrating in  $r$  shows that

$$u(\mu x) = \mu^{-\frac{4+\alpha+\beta}{p-1}} u(x), \quad \forall \mu \in (0, 1], \quad x \in B_R \setminus \{0\}. \quad \square$$

**Remark 2.2.** From the proof of Theorem 2.1, we can find that if the linear combination of Pohozaev identity and the identity (2.16) minus some terms of  $\mathfrak{T}_1$  and  $\mathfrak{T}_2$ , then it is equivalent to the derivative form of the monotonicity formula (2.14).

### 3 Proof of Theorem 1.1

First, we give the expression of  $N_{\alpha, \beta}(p)$ . Now, we define four functions by

$$\begin{aligned} \mathfrak{f}(N) &:= p \frac{4+\alpha+\beta}{p-1} \left( \frac{4+\alpha+\beta p}{p-1} + 2 \right) \left( N - 2 - \frac{4+\alpha+\beta}{p-1} \right) \left( N - 4 - \frac{4+\alpha+\beta p}{p-1} \right), \\ \mathfrak{g}(N) &:= p \left( \frac{4+\alpha+\beta p}{p-1} + 2 \right) \left( N - 4 - \frac{4+\alpha+\beta p}{p-1} \right) + p \frac{4+\alpha+\beta}{p-1} \left( N - 2 - \frac{4+\alpha+\beta}{p-1} \right), \\ \mathfrak{F}(N) &:= \frac{(N+\beta)^2(N-4-\beta)^2}{16}, \\ \mathfrak{G}(N) &:= \frac{(N+\beta)(N-4-\beta)}{2}. \end{aligned}$$

Differentiating the functions  $\mathfrak{f}(N)$  and  $\mathfrak{F}(N)$  in  $N$ , we obtain

$$\begin{aligned} \mathfrak{f}'(N) &= p \frac{4+\alpha+\beta}{p-1} \left( 2 + \frac{4+\alpha+\beta p}{p-1} \right) \left( 2N - 6 - \beta - \frac{8+2\alpha+2\beta}{p-1} \right), \\ \mathfrak{F}'(N) &= \frac{1}{4} (N+\beta)(N-2)(N-4-\beta). \end{aligned}$$

A simple computation yields

$$\mathfrak{f}(4 + \beta + 2\lambda) - \mathfrak{F}(4 + \beta + 2\lambda) = (p - 1)\lambda^2(2 + \beta + \lambda)^2 > 0,$$

$$\begin{aligned} & \mathfrak{f}(4 + \beta + (4p + 1)\lambda) - \mathfrak{F}(4 + \beta + (4p + 1)\lambda) \\ &= 4p^2\lambda^2(2 + \beta + \lambda)(2 + \beta + 4p\lambda) - \frac{(4p + 1)^2\lambda^2}{16} \left[ (4 + 2\beta) + (4p + 1)\lambda \right]^2 \\ &< 0, \end{aligned}$$

$$\mathfrak{g}(4 + \beta + 2\lambda) - \mathfrak{G}(4 + \beta + 2\lambda) = 2(p - 1)\lambda(2 + \beta + \lambda) > 0,$$

$$\mathfrak{f}'(4 + \beta + 2\lambda) - \mathfrak{F}'(4 + \beta + 2\lambda) = (p - 1)\lambda(2 + \beta + \lambda)(2 + \beta + 2\lambda) > 0.$$

Therefore, we take the least real root  $N(p, \alpha, \beta)$  of the following algebra equation between  $4 + \beta + 2\lambda$  with  $4 + \beta + (4p + 1)\lambda$

$$\begin{aligned} & (p^4 - 4p^3 + 6p^2 - 4p + 1)y^4 - (8p^4 - 32p^3 + 48p^2 - 32p + 8)y^3 \\ & - (p^2 - 2p + 1) \left[ (32\alpha + 104\beta + 16\alpha\beta + 18\beta^2 + 112)p^2 + (16\alpha\beta \right. \\ & + 16\alpha^2 - 4\beta^2 + 16\beta + 96\alpha + 160)p + 8\beta + 2\beta^2 - 16 \left. \right] y^2 \\ & + \left\{ \left[ (48 + 44\beta + 12\beta^2 + 8\alpha\beta + 12\alpha + \alpha\beta^2 + \beta^3)p^2 + (64 + 56\alpha + 28\alpha\beta \right. \right. \\ & + 10\alpha^2 + 40\beta + 10\beta^2 + 4\alpha\beta^2 + 3\alpha^2\beta + \beta^3) p + 28\alpha + 16 + 14\alpha^2 + 2\alpha^3 \\ & + 12\beta + 12\alpha\beta + 3\alpha^2\beta + 2\beta^2 + \alpha\beta^2 \left. \right] 16(p^2 - p) - (p - 1)^4(32\beta + 8\beta^2) \left. \right\} y \\ & + (p - 1)^4\beta^2(\beta + 4)^2 - 16 \left[ (8 + (2 + \beta)\alpha + (6 + \beta)\beta)p + (6 + \alpha + \beta)\alpha + 2\beta + 8 \right] \\ & \times \left[ (8 + 2\beta)p^2 + (6\beta + 6\alpha + \alpha\beta + \beta^2 + 8)p + 2\alpha + \alpha^2 + \alpha\beta \right] p \\ & = 0. \end{aligned}$$

Define

$$N_{\alpha, \beta}(p) := N(p, \alpha, \beta). \quad (3.1)$$

Then, for any  $4 + \beta + \frac{8 + 2\alpha + 2\beta}{p - 1} < N < N_{\alpha, \beta}(p)$ , we find

$$\mathfrak{f}(N) > \frac{(N + \beta)^2(N - 4 - \beta)^2}{16}. \quad (3.2)$$

Furthermore, combining the above inequality with the inequality  $a + b \geq 2\sqrt{ab}$ , for all  $a, b \geq 0$ , we have

$$\mathfrak{g}(N) > \frac{(N + \beta)(N - 4 - \beta)}{2}. \quad (3.3)$$

On the other hand, we easily check that the equality  $\mathfrak{f}(N) - \mathfrak{F}(N) > 0$  holds, if one of the following conditions holds:

- (i)  $\alpha = \beta$  and  $4 + \alpha + \frac{8 + 4\alpha}{p-1} < N < 8 + 3\alpha + \frac{8 + 4\alpha}{p-1}$ ; or
- (ii)  $\alpha = \beta = 0$  and  $4 + \frac{8}{p-1} < N < 2 + \frac{4(p+1)}{p-1} \left( \sqrt{\frac{2p}{p+1}} + \sqrt{\frac{2p}{p+1} - \sqrt{\frac{2p}{p+1}}} \right)$ .

Let us recall that if we take

$$\Gamma = \frac{4 + \alpha + \beta}{p-1} \left( \frac{4 + \alpha + \beta p}{p-1} + 2 \right) \left( N - 2 - \frac{4 + \alpha + \beta}{p-1} \right) \left( N - 4 - \frac{4 + \alpha + \beta p}{p-1} \right),$$

then

$$u_\Gamma(r) = \Gamma^{\frac{1}{p-1}} r^{-\frac{4+\alpha+\beta}{p-1}}$$

is a singular solution of (1.1) in  $\mathbb{R}^N \setminus \{0\}$ . By the well-known weighted Hardy-Rellich inequality ([15]) with the best constant

$$\int_{\mathbb{R}^N} \frac{|\Delta \psi|^2}{|x|^\beta} dx \geq \frac{(N + \beta)^2 (N - 4 - \beta)^2}{16} \int_{\mathbb{R}^N} \frac{\psi^2}{|x|^{4+\beta}} dx, \quad \forall \psi \in H_{loc}^2(\mathbb{R}^N),$$

we conclude that the singular solution  $u_\Gamma$  is stable in  $\mathbb{R}^N \setminus \{0\}$  if and only if

$$\mathfrak{f}(N) = p\Gamma \leq \frac{(N + \beta)^2 (N - 4 - \beta)^2}{16} = \mathfrak{F}(N).$$

Here  $-1 - \frac{\sqrt{1 + (N-1)^2}}{2} \leq \beta \leq \frac{N-4}{2}$ .

**Proof of Theorem 1.1.** Since  $u \in W^{2,2}(B_2 \setminus B_1)$ ,  $|x|^\alpha |u|^{p+1} \in L_{loc}^1(\mathbb{R}^N \setminus \{0\})$  and  $|x|^{-\beta} |\Delta u|^2 \in L_{loc}^1(\mathbb{R}^N \setminus \{0\})$ , we can assume that there exists a  $\Psi \in W^{2,2}(\mathbb{S}^{N-1}) \cap L^{p+1}(\mathbb{S}^{N-1})$ , such that in polar coordinates

$$u(r, \theta) = r^{-\frac{4+\alpha+\beta}{p-1}} \Psi(\theta).$$

Substituting into (1.1) to get

$$\Delta_\theta^2 \Psi - \Upsilon \Delta_\theta \Psi + \Gamma \Psi = |\Psi|^{p-1} \Psi,$$

where

$$\begin{aligned} \Upsilon &= \lambda(N - 2 - \lambda) + \left( \frac{4 + \alpha + \beta p}{p-1} + 2 \right) \left( N - 4 - \frac{4 + \alpha + \beta p}{p-1} \right), \\ \Gamma &= \lambda(N - 2 - \lambda) \left( \frac{4 + \alpha + \beta p}{p-1} + 2 \right) \left( N - 4 - \frac{4 + \alpha + \beta p}{p-1} \right). \end{aligned}$$

Multiplying the above equation by  $\Psi$  and integration by parts yields

$$\int_{\mathbb{S}^{N-1}} |\Delta_\theta \Psi|^2 + \Upsilon |\nabla_\theta \Psi|^2 + \Gamma \Psi^2 = \int_{\mathbb{S}^{N-1}} |\Psi|^{p+1}. \quad (3.4)$$

Since  $u$  is a stable solution, we can take a test function  $r^{-\frac{N-4-\beta}{2}} \Psi(\theta) \xi_\varepsilon(r)$  and obtain

$$p \int_{\mathbb{R}^N} |x|^\alpha |u|^{p-1} \left( r^{-\frac{N-4-\beta}{2}} \Psi(\theta) \xi_\varepsilon(r) \right)^2 dx \leq \int_{\mathbb{R}^N} \frac{\left| \Delta \left( r^{-\frac{N-4-\beta}{2}} \Psi(\theta) \xi_\varepsilon(r) \right) \right|^2}{|x|^\beta} dx. \quad (3.5)$$

Here, for any  $\varepsilon > 0$ , we choose  $\xi_\varepsilon \in C_0^2\left(\left(\frac{\varepsilon}{3}, \frac{3}{\varepsilon}\right)\right)$  such that  $\xi_\varepsilon \equiv 1$  in  $(\varepsilon, \frac{1}{\varepsilon})$  and

$$r|\xi'_\varepsilon(r)| + r^2|\xi''_\varepsilon(r)| \leq C,$$

for all  $r > 0$ . Then one can easily deduce that

$$\int_0^\infty r^{-1} \xi_\varepsilon^2(r) dr \geq \int_\varepsilon^{\frac{1}{\varepsilon}} r^{-1} dr = 2|\ln \varepsilon|,$$

and

$$\int_0^\infty \left[ r|\xi'_\varepsilon(r)|^2 + r^3|\xi''_\varepsilon(r)|^2 + |\xi'_\varepsilon(r)\xi_\varepsilon(r)| + r|\xi_\varepsilon(r)\xi''_\varepsilon(r)| \right] dr \leq C.$$

Applying the coordinate transformation to the left hand side of (3.5), we get

$$\begin{aligned} p \int_0^{+\infty} \int_{\mathbb{S}^{N-1}} r^\alpha |u|^{p-1} \left( r^{-\frac{N-4-\beta}{2}} \Psi(\theta) \xi_\varepsilon(r) \right)^2 r^{N-1} dr d\theta \\ = p \left( \int_{\mathbb{S}^{N-1}} |\Psi|^{p+1} d\theta \right) \left( \int_0^{+\infty} r^{-1} \xi_\varepsilon^2(r) dr \right). \end{aligned} \quad (3.6)$$

A direct calculation finds

$$\begin{aligned} \Delta \left( r^{-\frac{N-4-\beta}{2}} \Psi(\theta) \xi_\varepsilon(r) \right) = & -\frac{(N+\beta)(N-4-\beta)}{4} r^{-\frac{N-\beta}{2}} \xi_\varepsilon(r) \Psi(\theta) + r^{-\frac{N-\beta}{2}} \xi_\varepsilon(r) \Delta_\theta \Psi \\ & + (3+\beta) r^{-\frac{N-2-\beta}{2}} \xi'_\varepsilon(r) \Psi(\theta) + r^{-\frac{N-4-\beta}{2}} \xi''_\varepsilon(r) \Psi(\theta), \end{aligned}$$

and inserting into the right hand side of (3.5) yields

$$\begin{aligned} & \int_{\mathbb{R}^N} |x|^{-\beta} \left| \Delta \left( r^{-\frac{N-4-\beta}{2}} \Psi(\theta) \xi_\varepsilon(r) \right) \right|^2 dx \\ & \leq \left[ \int_{\mathbb{S}^{N-1}} \left( |\Delta_\theta \Psi|^2 + \frac{(N+\beta)(N-4-\beta)}{2} |\nabla_\theta \Psi|^2 + \frac{(N+\beta)^2(N-4-\beta)^2}{16} \Psi^2 \right) d\theta \right] \\ & \quad \times \left( \int_0^{+\infty} r^{-1} \xi_\varepsilon^2(r) dr \right) \\ & \quad + O \left\{ \int_0^{+\infty} \left[ r|\xi'_\varepsilon(r)|^2 + r^3|\xi''_\varepsilon(r)|^2 + |\xi'_\varepsilon(r)\xi_\varepsilon(r)| + r\xi_\varepsilon(r)|\xi''_\varepsilon(r)| \right] dr \right\} \\ & \quad \times \int_{\mathbb{S}^{N-1}} [\Psi(\theta)^2 + |\nabla_\theta \Psi(\theta)|^2] d\theta. \end{aligned} \quad (3.7)$$

Put (3.6) and (3.7) back into (3.5), take  $\varepsilon \rightarrow 0$ , and pass to the limit to obtain

$$\begin{aligned} & p \int_{\mathbb{S}^{N-1}} |\Psi|^{p+1} d\theta \\ & \leq \int_{\mathbb{S}^{N-1}} \left[ |\Delta_\theta \Psi|^2 + \frac{(N+\beta)(N-4-\beta)}{2} |\nabla_\theta \Psi|^2 + \frac{(N+\beta)^2(N-4-\beta)^2}{16} \Psi^2 \right] d\theta. \end{aligned}$$

Now, combining the above inequality with (3.4), we have

$$\begin{aligned} & \int_{\mathbb{S}^{N-1}} (p-1) |\Delta_\theta \Psi|^2 + \left( p\Upsilon - \frac{(N+\beta)(N-4-\beta)}{2} \right) |\nabla_\theta \Psi|^2 \\ & \quad + \left( p\Gamma - \frac{(N+\beta)^2(N-4-\beta)^2}{16} \right) \Psi^2 \leq 0. \end{aligned}$$

Since  $4 + \beta + \frac{8+2\alpha+2\beta}{p-1} < N < N_{\alpha,\beta}(p)$ , it implies from the definition of  $N_{\alpha,\beta}(p)$ , (3.2) and (3.3) that

$$\Psi(\theta) \equiv 0.$$

Therefore, we get

$$u \equiv 0. \quad \square$$

## 4 Proof of Theorem 1.2

**Proof of Theorem 1.2.** We divide the proof into three cases.

**Case I.**  $5 \leq N < 4 + \beta + \frac{8+2\alpha+2\beta}{p-1}$ .

Since  $N < 4 + \beta + \frac{8+2\alpha+2\beta}{p-1}$ , it implies from (2.4) that as  $R \rightarrow +\infty$ ,

$$\int_{B_R(x)} \left[ \frac{|\Delta u|^2}{|z|^\beta} + |z|^\alpha |u|^{p+1} \right] dz \leq CR^{N-4-\beta-\frac{8+2\alpha+2\beta}{p-1}} \rightarrow 0.$$

Therefore, we get

$$u \equiv 0.$$

**Case II.**  $N = 4 + \beta + \frac{8+2\alpha+2\beta}{p-1}$ .

From the inequality (2.4), we obtain that

$$\int_{\mathbb{R}^N} \left[ \frac{|\Delta u|^2}{|z|^\beta} + |z|^\alpha |u|^{p+1} \right] dz < +\infty,$$

implies

$$\lim_{R \rightarrow +\infty} \int_{\mathfrak{D}} \left[ \frac{|\Delta u|^2}{|z|^\beta} + |z|^\alpha |u|^{p+1} \right] dz = 0,$$

where  $\mathfrak{D} := B_{2R}(x) \setminus B_R(x)$ . Applying (2.3) and Hölder's inequality yields

$$\begin{aligned}
\int_{B_R(x)} \left[ \frac{|\Delta u|^2}{|z|^\beta} + |z|^\alpha |u|^{p+1} \right] dz &\leq CR^{-2} \int_{\mathfrak{D}} \frac{|u \Delta u|}{|z|^\beta} dz + CR^{-4} \int_{\mathfrak{D}} \frac{u^2}{|z|^\beta} dz \\
&\leq C\mathfrak{C}R^{-2} \left( \int_{\mathfrak{D}} |z|^\alpha |u|^{p+1} dz \right)^{\frac{1}{p+1}} \left( \int_{\mathfrak{D}} |z|^{-\frac{2\alpha+\beta(p+1)}{p-1}} dz \right)^{\frac{p-1}{2(p+1)}} \\
&\quad + CR^{-4} \left( \int_{\mathfrak{D}} |z|^\alpha |u|^{p+1} dz \right)^{\frac{2}{p+1}} \left( \int_{\mathfrak{D}} |z|^{-\frac{2\alpha+\beta(p+1)}{p-1}} dz \right)^{\frac{p-1}{p+1}} \\
&\leq C\mathfrak{C}R^{\left[ N-4-\beta-\frac{8+2\alpha+2\beta}{p-1} \right] \frac{p-1}{2(p+1)}} \left( \int_{\mathfrak{D}} |z|^\alpha |u|^{p+1} dz \right)^{\frac{p-1}{2(p+1)}} \\
&\quad + CR^{\left[ N-4-\beta-\frac{8+2\alpha+2\beta}{p-1} \right] \frac{p-1}{p+1}} \left( \int_{\mathfrak{D}} |z|^\alpha |u|^{p+1} dz \right)^{\frac{p-1}{p+1}},
\end{aligned}$$

where  $\mathfrak{C} = \left( \int_{\mathfrak{D}} \frac{|\Delta u|^2}{|z|^\beta} dz \right)^{\frac{1}{2}}$ . From  $N = 4 + \beta + \frac{8+2\alpha+2\beta}{p-1}$ , it implies that the right hand side of the above inequality converges to 0 as  $R \rightarrow +\infty$ . Therefore, we obtain

$$u \equiv 0.$$

**Case III.**  $4 + \beta + \frac{8+2\alpha+2\beta}{p-1} < N < N_{\alpha,\beta}(p)$ .

First, we will obtain some properties of the function  $\mathcal{M}$ .

**Lemma 4.1.**  $\lim_{r \rightarrow +\infty} \mathcal{M}(r; 0, u) < +\infty$ .

*Proof.* The proof mainly use the estimate (2.4) and the monotonicity of the function  $\mathcal{M}(r; 0, u)$  in  $r$ .

Applying (2.4) to estimate the first term in the right hand side of (2.14) yields

$$\begin{aligned}
&r^{\frac{8+2\alpha+2\beta}{p-1}+4+\beta-N} \int_{B_r} \left[ \frac{1}{2} \frac{(\Delta u)^2}{|x|^\beta} - \frac{1}{p+1} |x|^\alpha |u|^{p+1} \right] dx \\
&\leq Cr^{\frac{8+2\alpha+2\beta}{p-1}+4+\beta-N} r^{N-4-\beta-\frac{8+2\alpha+2\beta}{p-1}} \\
&\leq C.
\end{aligned}$$

Utilize Hölder's inequality to estimate the second term in the right hand side of (2.14)

$$\begin{aligned}
&r^{\frac{8+2\alpha+2\beta}{p-1}+1-N} \int_{\partial B_r} u^2 \leq \frac{1}{r} \int_r^{2r} \left( \mu^{\frac{8+2\alpha+2\beta}{p-1}+1-N} \int_{\partial B_\mu} u^2 dS \right) d\mu \\
&\leq \frac{1}{r} \left( \int_{B_{2r} \setminus B_r} \left( |x|^{\frac{8+2\alpha+2\beta}{p-1}+1-N-\frac{2\alpha}{p+1}} \right)^{\frac{p+1}{p-1}} \left( \int_{B_{3r}} |x|^\alpha |u|^{p+1} \right)^{\frac{2}{p+1}} \right. \\
&\leq Cr^{\left[ \frac{8+2\alpha+2\beta}{p-1}-N-\frac{2\alpha}{p+1}+N\frac{p-1}{p+1} \right] \frac{2}{p+1}} r^{\frac{2}{p+1} \left[ N-4-\beta-\frac{8+2\alpha+2\beta}{p-1} \right]} \\
&\leq C.
\end{aligned}$$

Similarly, we find

$$\begin{aligned} \frac{d}{dr} \left( r^{2\lambda+2-N} \int_{\partial B_r} u^2 \right) &\leq \frac{1}{r^2} \int_r^{2r} \int_{\iota}^{\iota+r} \frac{d}{d\mu} \left( \mu^{2\lambda+2-N} \int_{\partial B_\mu} u^2 \right) d\mu d\iota \\ &\leq C. \end{aligned}$$

By the interpolation inequality and Hölder's inequality, we get

$$\begin{aligned} \int_{B_r} |\nabla u|^2 &\leq Cr^2 \int_{B_r} |\Delta u|^2 + Cr^{-2} \int_{B_r} u^2 \\ &\leq Cr^2 \int_{B_r} |x|^\beta \frac{|\Delta u|^2}{|x|^\beta} + Cr^{-2} \left( \int_{B_r} |x|^\alpha |u|^{p+1} \right)^{\frac{2}{p+1}} \left( \int_{B_r} |x|^{-\frac{2\alpha}{p-1}} dx \right)^{\frac{p-1}{p+1}} \\ &\leq Cr^{N-2-\frac{8+2\alpha+2\beta}{p-1}}. \end{aligned} \tag{4.1}$$

Then, it implies that

$$r^{\frac{8+2\alpha+2\beta}{p-1}+3-N} \int_{\partial B_r} |\nabla u|^2 dS \leq \frac{1}{r} \int_r^{2r} \left( \mu^{\frac{8+2\alpha+2\beta}{p-1}+3-N} \int_{\partial B_\mu} |\nabla u|^2 dS \right) d\mu \leq C.$$

Therefore, we get the boundedness of the fifth and sixth terms in the right hand side of (2.14). Utilizing Hölder's inequality and (4.1), we find

$$\begin{aligned} &\frac{1}{r^2} \int_r^{2r} \int_{\iota}^{\iota+r} \frac{\mu^3}{2} \frac{d}{d\mu} \left[ \mu^{2\lambda+1-N} \int_{\partial B_\mu} \left( \lambda \mu^{-1} u + \frac{\partial u}{\partial r} \right)^2 \right] d\mu d\iota \\ &= \frac{1}{2r^2} \int_r^{2r} \left\{ (\iota+r)^{2\lambda+4-N} \int_{\partial B_{\iota+r}} \left[ \lambda (\iota+r)^{-1} u + \frac{\partial u}{\partial r} \right]^2 - \iota^{2\lambda+4-N} \int_{\partial B_\iota} \left[ \lambda \iota^{-1} u + \frac{\partial u}{\partial r} \right]^2 \right\} \\ &\quad - \frac{3}{2r^2} \int_r^{2r} \int_{\iota}^{\iota+r} \mu^{2\lambda+3-N} \int_{\partial B_\mu} \left( \lambda \mu^{-1} u + \frac{\partial u}{\partial r} \right)^2 \\ &\leq \frac{C}{r^2} \int_{B_{3r} \setminus B_r} |x|^{2\lambda+2-N} \left( u^2 + |x|^2 \left( \frac{\partial u}{\partial r} \right)^2 \right) dx \\ &\leq C. \end{aligned}$$

Consequently, we obtain the desired result.  $\square$

**Lemma 4.2.** *For all  $\kappa > 0$ , define blowing down sequences*

$$u^\kappa(x) := \kappa^{\frac{4+\alpha+\beta}{p-1}} u(\kappa x),$$

*then  $u^\kappa$  strongly converges to  $u^\infty$  in  $W_{loc}^{1,2}(\mathbb{R}^N) \cap L_{loc}^{p+1}(\mathbb{R}^N)$ . Furthermore,  $u^\infty$  is a homogeneous stable solution of (1.1).*

*Proof.* Since  $u$  is a stable solution of (1.1), we can find

$$\begin{aligned}
p \int_{\mathbb{R}^N} |x|^\alpha |u^\kappa|^{p-1} \zeta^2(x) dx &= p \int_{\mathbb{R}^N} |\kappa x|^\alpha \kappa^{4+\beta} |u(\kappa x)|^{p-1} \zeta^2(x) dx \\
&= p \kappa^{4+\beta-N} \int_{\mathbb{R}^N} |y|^\alpha |u(y)|^{p-1} \psi^2(y) dy \quad \text{taking } \psi(y) := \zeta(x), x = \frac{y}{\kappa} \\
&\leq \kappa^{4+\beta-N} \int_{\mathbb{R}^N} \frac{|\Delta \psi(y)|^2}{|y|^\beta} dy \\
&= \int_{\mathbb{R}^N} \frac{|\Delta \zeta|^2}{|x|^\beta} dx.
\end{aligned} \tag{4.2}$$

Thus,  $u^\kappa$  is a stable solution of (1.1). Furthermore, from (2.4), it implies that

$$\begin{aligned}
&\int_{B_r(x)} \left[ |y|^{-\beta} (\Delta u^\kappa)^2 + |y|^\alpha |u^\kappa|^{p+1} \right] dy \\
&= \kappa^{4+\beta+\frac{8+2\alpha+2\beta}{p-1}-N} \int_{B_{\kappa r}(x)} \left[ |z|^{-\beta} |\Delta(z)|^2 + |z|^\alpha |u(z)|^{p+1} \right] dz \\
&\leq C r^{N-4-\beta-\frac{8+2\alpha+2\beta}{p-1}},
\end{aligned}$$

and applying Hölder's inequality yields

$$\begin{aligned}
\int_{B_r(x)} |u^\kappa|^2 dz &\leq \left( \int_{B_r(x)} |z|^\alpha |u^\kappa|^{p+1} dz \right)^{\frac{2}{p+1}} \left( \int_{B_r(x)} |z|^{-\frac{2\alpha}{p-1}} dz \right)^{\frac{p-1}{p+1}} \\
&\leq C r^{N-2\lambda}.
\end{aligned}$$

Clearly, we also obtain

$$\begin{aligned}
\int_{B_r(x)} |\Delta u^\kappa|^2 dz &= \int_{B_{\kappa r}(x)} \kappa^{2\lambda+4-N} |z|^\beta \frac{|\Delta u(z)|^2}{|z|^\beta} dz \\
&\leq C r^{N-4-2\lambda}.
\end{aligned}$$

By the application of the elliptic regularity theory, it implies that  $u^\kappa$  are uniformly bounded in  $W_{loc}^{2,2}(\mathbb{R}^N)$ . Again  $u \in C^4(\mathbb{R}^N)$  implies  $u^\kappa \in L_{loc}^{p+1}(\mathbb{R}^N)$ . Then we can suppose that  $u^\kappa \rightharpoonup u^\infty$  weakly in  $W_{loc}^{2,2}(\mathbb{R}^N) \cap L_{loc}^{p+1}(\mathbb{R}^N)$  (if necessary, we can extract a subsequence). Now, using the standard embeddings, we get  $u^\kappa \rightarrow u^\infty$  strongly in  $W_{loc}^{1,2}(\mathbb{R}^N)$ . Therefore, applying the interpolation inequality between  $L^q$  spaces with  $q \in (1, p+1)$ , we get that, for any ball  $B_r$

$$\|u^\kappa - u^\infty\|_{L^q(B_r)} \leq \|u^\kappa - u^\infty\|_{L^1(B_r)}^t \|u^\kappa - u^\infty\|_{L^{p+1}(B_r)}^{1-t} \rightarrow 0, \quad \text{as } \kappa \rightarrow +\infty, \tag{4.3}$$

where  $t \in (0, 1)$  satisfying  $\frac{1}{q} = t + \frac{1-t}{p+1}$ . Next, combining with the definition of  $u^\kappa$  and



(4.2), we conclude that, for any  $\zeta \in C_0^2(\mathbb{R}^N)$

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{\Delta u^\infty}{|x|^\beta} \Delta \zeta - |x|^\alpha |u^\infty|^{p-1} u^\infty \zeta &= \lim_{\kappa \rightarrow \infty} \int_{\mathbb{R}^N} \frac{\Delta u^\kappa}{|x|^\beta} \Delta \zeta - |x|^\alpha |u^\kappa|^{p-1} u^\kappa \zeta, \\ \int_{\mathbb{R}^N} \frac{(\Delta \zeta)^2}{|x|^\beta} - p |x|^\alpha |u^\infty|^{p-1} \zeta^2 &= \lim_{\kappa \rightarrow \infty} \int_{\mathbb{R}^N} \frac{(\Delta \zeta)^2}{|x|^\beta} - p |x|^\alpha |u^\kappa|^{p-1} \zeta^2 \geq 0, \end{aligned}$$

that is,  $u^\infty \in W_{loc}^{2,2}(\mathbb{R}^N) \cap L_{loc}^{p+1}(\mathbb{R}^N)$  is a stable solution of (1.1) in  $\mathbb{R}^N$ .

From the boundedness and monotonicity of  $\mathcal{M}(r; 0, u)$ , it implies that for any  $0 < r_1 < r_2 < +\infty$ ,

$$\lim_{\kappa \rightarrow \infty} [\mathcal{M}(\kappa r_2; 0, u) - \mathcal{M}(\kappa r_1; 0, u)] = 0.$$

Again using the scaling invariance and Theorem 2.1, we get

$$\begin{aligned} 0 &= \lim_{\kappa \rightarrow \infty} [\mathcal{M}(r_2; 0, u^\kappa) - \mathcal{M}(r_1; 0, u^\kappa)] \\ &= \lim_{\kappa \rightarrow \infty} \int_{r_1}^{r_2} \frac{d}{d\mu} \mathcal{M}(\mu; 0, u^\kappa) d\mu \\ &\geq C(N, p, \alpha, \beta) \int_{B_{r_2} \setminus B_{r_1}} |x|^{2+2\lambda-N} \left( \lambda \mu^{-1} u^\infty + \frac{\partial u^\infty}{\partial \mu} \right)^2 dx. \end{aligned}$$

Adopting the same calculation as Theorem 2.1, we obtain that  $u^\infty$  is homogeneous.  $\square$

**Lemma 4.3.**  $\lim_{r \rightarrow \infty} \mathcal{M}(r; 0, u) = 0.$

*Proof.* Since  $u^\infty$  is a homogeneous, stable solution of (1.1), it implies from Theorem 1.1 that

$$u^\infty \equiv 0.$$

Combining (4.3) with the above equality, we find that

$$\lim_{\kappa \rightarrow +\infty} u^\kappa = 0, \text{ strongly in } L^2(B_6),$$

i.e.,

$$\lim_{\kappa \rightarrow +\infty} \int_{B_6} |u^\kappa|^2 = 0.$$

From the uniform boundedness of  $\Delta u^\kappa$  in  $L^2(B_6)$ , we get

$$\lim_{\kappa \rightarrow \infty} \int_{B_6} |u^\kappa \Delta u^\kappa| \leq \lim_{\kappa \rightarrow \infty} \left( \int_{B_6} |u^\kappa|^2 \right)^{\frac{1}{2}} \left( \int_{B_6} |\Delta u^\kappa|^2 \right)^{\frac{1}{2}} = 0.$$

Therefore, it implies from (2.3) that

$$\lim_{\kappa \rightarrow +\infty} \int_{B_1} |\Delta u^\kappa|^2 + |x|^\alpha |u^\kappa|^{p+1} \leq C \lim_{\kappa \rightarrow +\infty} \int_{B_6} |u^\kappa|^2 + |u^\kappa \Delta u^\kappa| = 0.$$

A direct application of the interior  $L^p$ -estimates gets

$$\lim_{\kappa \rightarrow +\infty} \int_{B_2} \sum_{j \leq 2} |\nabla^j u^\kappa| = 0,$$

implies

$$\int_1^2 \left( \sum_{i=1}^{\infty} \int_{\partial B_r} \sum_{j \leq 2} |\nabla^j u^{\kappa_i}|^2 \right) dr \leq \sum_{i=1}^{\infty} \int_{B_{2r} \setminus B_r} \sum_{j \leq 2} |\nabla^j u^{\kappa_i}|^2 \leq 1.$$

Then, let us note that there exists a  $\gamma \in (1, 2)$  such that

$$\lim_{\kappa \rightarrow \infty} \|u^\kappa\|_{W^{2,2}(\partial B_\gamma)} = 0.$$

Now, combining the above results with the scaling invariance of  $\mathcal{M}(r; 0, u)$ , we obtain

$$\lim_{i \rightarrow \infty} \mathcal{M}(\kappa_i \gamma; 0, u) = \lim_{i \rightarrow \infty} \mathcal{M}(\gamma; 0, u^{\kappa_i}) = 0.$$

Again since  $\kappa_i \gamma \rightarrow +\infty$  and  $\mathcal{M}(r; 0, u)$  is non-decreasing in  $r$ , we get

$$\lim_{r \rightarrow \infty} \mathcal{M}(r; 0, u) = 0. \quad \square$$

Since  $u \in C^4(\mathbb{R}^N)$ , we get  $\lim_{r \rightarrow 0} \mathcal{M}(r; 0, u) = 0$ . Again using the monotonicity of  $\mathcal{M}(r; 0, u)$  and Lemma 4.3, we get

$$\mathcal{M}(r; 0, u) = 0, \quad \text{for all } r > 0.$$

Therefore, combining with Theorem 2.1, we conclude that  $u$  is homogeneous and by Theorem 1.1

$$u \equiv 0. \quad \square$$

**Acknowledge:** The author wishes to express his warmest thanks to Chern Institute of Mathematics, Nankai University (where part of this work was done) for their warm hospitality.

## References

- [1] A. Bahri, P.L. Lions, Solutions of superlinear elliptic equations and their Morse indices, Comm. Pure Appl. Math. 45 (1992) 1205-1215;
- [2] M. Bidaut-Véron, L. Véron, Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations, Invent. Math. 106 (1991) 489-539;

- [3] C. Cowan, Liouville theorems for stable Lane-Emden systems with biharmonic problems, *Nonlinearity* 26 (2013) 2357-2371;
- [4] E.N. Dancer, Stable and finite Morse index solutions on  $\mathbb{R}^N$  or on bounded domains with small diffusion, II, *Indiana Univ. Math. J.* 53 (2004) 97-108;
- [5] E.N. Dancer, Stable and finite Morse index solutions on  $\mathbb{R}^N$  or on bounded domains with small diffusion, *Trans. Amer. Math. Soc.* 357 (2005) 1225-1243;
- [6] E.N. Dancer, Finite Morse index solutions of supercritical problems, *J. Reine Angew. Math.* 620 (2008) 213-233;
- [7] E.N. Dancer, Y.H. Du, Z.M. Guo, Finite Morse index solutions of an elliptic equation with supercritical exponent, *J. Differential Equations* 250 (2011) 3281-3310;
- [8] J. Dávila, L. Dupaigne, A. Farina, Partial regularity of finite Morse index solutions to the Lane-Emden equation, *J. Funct. Anal.* 261 (2011) 218-232;
- [9] J. Dávila, L. Dupaigne, K.L. Wang, J.C. Wei, A monotonicity formula and a Liouville-type theorem for a fourth order supercritical problem, *Adv. Math.* 258 (2014) 240-285;
- [10] Y.H. Du, Z.M. Guo, K.L. Wang, Monotonicity formula and  $\varepsilon$ -regularity of stable solutions to supercritical problems and applications to finite Morse index solutions, *Calc. Var. PDEs* DOI 10.1007/s00526-013-0649-X;
- [11] P. Esposito, Linear instability of entire solutions for a class of non-autonomous elliptic equations, *Proc. Roy. Soc. Edinburgh Sect. A* 138 (2008) 1005-1018;
- [12] M. Fazly, Liouville type theorems for stable solutions of certain elliptic systems, *Adv. Nonlinear Stud.* 12 (2012) 1-17;
- [13] M. Fazly, N. Ghoussoub, On the Hénon-Lane-Emden conjecture, *Discrete Contin. Dyn. Syst.* 34 (2014) 2513-2533;
- [14] A. Farina, On the classification of solutions of Lane-Emden equation on unbounded domains of  $\mathbb{R}^N$ , *J. Math. Pures Appl.* 87 (2007) 537-561;
- [15] N. Ghoussoub, A. Moradifard, Bessel pairs and optimal Hardy and Hardy-Rellich inequalities, *Math. Ann.* 349 (2011) 1-57;
- [16] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* 24 (1981) 525-598;
- [17] L.G. Hu, Liouville-type theorems for the fourth order nonlinear elliptic equation, *J. Differential Equations* 256 (2014) 1817-1846;

- [18] E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic systems in  $\mathbb{R}^N$ , *Diff. Integral Equ.* 9 (1996) 465-479;
- [19] M. Montenegro, Minimal solutions for a class of elliptic systems, *Bull. London Math. Soc.* 37 (2005) 405-416;
- [20] F. Pacard, Partial regularity for weak solutions of a nonlinear elliptic equations, *Manuscripta Math.* 79 (1993) 161-172;
- [21] F. Pacard, Convergence and partial regularity for weak solutions of some nonlinear elliptic equation: the supercritical case, *Ann. Inst. Henri Poincaré* 11 (1994) 537-551;
- [22] Q.H. Phan, Liouville-type theorems and bounds of solutions for Hardy-Hénon elliptic systems, *Adv. Diff. Equ.* 17 (2012) 605-634;
- [23] Q.H. Phan, Ph. Souplet, Liouville-type theorems and bounds of solutions of Hardy-Hénon equations, *J. Differential Equations* 252 (2012) 2544-2562;
- [24] P. Poláčik, P. Quittner, Ph. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. elliptic equations and systems, *Duke Math. J.* 139 (2007) 555-579;
- [25] P. Pucci, J. Serrin, A general variational identity, *Indiana Univ. Math. J.* 35 (1986) 681-703;
- [26] J. Serrin, H. Zou, Non-existence of positive solutions of Lane-Emden systems, *Diff. Integral Equ.* 9 (1996) 635-653;
- [27] J. Serrin, H. Zou, Existence of positive solutions of the Lane-Emden system, *Atti Semin. Mat. Fis. Univ. Modena* 46 (1998) 369-380;
- [28] Ph. Souplet, The proof of the Lane-Emden conjecture in four space dimensions, *Adv. Math.* 221 (2009) 1409-1427;
- [29] C. Wang, D. Ye, Some Liouville theorems for Hénon type elliptic equations, *J. Funct. Anal.* 262 (2012) 1705-1727;
- [30] K.L. Wang, Partial regularity of stable solutions to the supercritical equations and its applications, *Nonlinear Anal.* 75 (2012) 5238-5260;
- [31] J.C. Wei, D. Ye, Liouville theorems for finite Morse index solutions of biharmonic problem, *Math. Ann.* 356 (2013) 1599-1612;
- [32] J.C. Wei, X.W. Xu, W. Yang, On the classification of stable solution to biharmonic problems in large dimensions, *Pacific J. Math.* 263 (2013) 495-512.